On Approximation of Double Barrier Option

ByoungSeon Choi* and Dongwoo Sheen†

Abstract

The trivariate joint probability density of Brownian motion and its maximum and minimum is well-known as an infinite series of Gaussian probability densities as well as a Fourier series. We present two formulas of the trivariate joint probability density using infinite products, which are computationally more efficient than the two infinite series solutions. Using the trivariate joint probability densities, we can price double barrier options under the Black-Scholes environment. Since the trivariate joint probability density is represented by an infinite series or an infinite product, we ought to use some approximate prices of a double barrier option. In this paper we present Gaussian series and Fourier series approximations of an Up-and-Out-Down-and-Out option, their error bounds, and stopping rules for approximations. Some numerical examples are presented to show usefulness of the approximations. Also, the results are compared to those by inverse Laplace transforms.

Keywords: Double barrier option, Error bound, Gaussian distribution, Fourier series, Laplace transform

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*Department of Economics, Seoul National University, Seoul, 151-742, Korea, (tel) +82-2-880-6394, (fax) +82-2-886-4231, (email) bschoi12@snu.ac.kr

†Department of Mathematics, Seoul National University, Seoul, 151-747, Korea, (tel) +82-2-880-6543, (fax) +82-2-887-4694, (email) dongwoosheen@gmail.com, Financial support from the Institute for Research in Finance and Economics of Seoul National University is gratefully acknowledged.

1 Introduction

A double-barrier is knocked either out or in, when the underlying touched a given lower bound or a given upper bound prior to expiry. Among various double barrier options, an Up-and-Out-Down-and-Out(UODO) option is the most popular for not only academics but also practitioners. Since the risk-neutral price of an UODO option under the Black-Scholes environment is represented by infinite sums, it is necessary in practice to approximate the risk-neutral price. In this paper we present some approximations of the risk-neutral price, their error bounds, and stopping rules for obtaining desirable approximations.

2 Trivariate joint probability density of Standard Brownian motion

Consider a standard Brownian motion $\{W_t | t \geq 0\}$ with $W_0 = 0$. Denote its maximum and minimum, respectively, by

$$M_t^W \doteq \max_{0 \leq s \leq t} W_s \quad \text{and} \quad m_t^W \doteq \min_{0 \leq s \leq t} W_s. \tag{2.1}$$

To price a double-barrier option analytically under the Black-Scholes environment, it is necessary to derive the trivariate joint distribution of $\left(W_t, m_t^W, M_t^W\right)$. In this section we derive it using several methods and compare the results in computational point of view.

2.1 Gaussian Series Solution

Define a stopping time by

$$\tau_c^W \doteq \inf \left\{u \geq 0 \;\mid\; W_u = c\right\}. \tag{2.2}$$

Clearly the mapping $c \mapsto \tau_c^W$ is increasing and left-continuous.

The following lemma is about the reflection principle of a Brownian motion, which is a basic tool to price a barrier option. See, *e.g.*, Shreve (2004, pp. 111-113).

Lemma 1. *For a standard Brownian motion $\{W_u \mid u \geq 0\}$, the following holds.*
(a) For any $c > 0$ and $h > 0$,

$$\Pr\left(M_t^W \geq c, W_t \geq c + h\right) = \Pr\left(M_t^W \geq c, W_t \leq c - h\right).$$

(b) For any $c > 0$,

$$\begin{aligned}\Pr\left(M_t^W \geq c\right) &= \Pr\left(\tau_c^W \leq t\right) = 2\Pr\left(W_t \geq c\right) \\ &= \Pr\left(\,|\,W_t\,| \,\geq c\right) = 2\left[1 - \int_c^\infty \phi(x; 0, t)\, dt\right],\end{aligned}$$

where $\phi\,(\cdot\,; m, v)$ denotes the normal probability density function with mean m and variance v.
(c) For any $t \geq 0$, the probability density function of τ_c^W is

$$f_{\tau_c^W}(t) \doteq \frac{|c|}{\sqrt{2\pi} t^{3/2}} \exp\left(-\frac{1}{2t} c^2\right).$$

(d) For any $\lambda \geq 0$, the Laplace transform of τ_c^W is

$$E\left(\exp\left(-\lambda \tau_c^W\right)\right) = \exp\left(-|c|\sqrt{2\lambda}\right).$$

(e) For any $M > 0$, $w \leq M$, and $t \geq 0$, the joint probability density function of $\left(M_t^W, W_t\right)$ is

$$f_{M_t^W, W_t}(w, M) \doteq [2M - w] \sqrt{\frac{2}{\pi t^3}} \exp\left(-\frac{[2M - w]^2}{2t}\right).$$

(f) For any $M > 0$, $w \leq M$, and $t \geq 0$, the conditional probability density function of M_t^W given $W_t = w$ is

$$f_{M_t^W \,|\, W_t}(M|w) \doteq \frac{2[2M - w]}{t} \exp\left(-\frac{2M[M - w]}{2t}\right).$$

□

Part (a) of Lemma 1 is called the reflection principle of a Brownian motion. Using it, we can derive the following lemma. See, *e.g.*, Freedman (1971, pp. 25-26).

Lemma 2. *Let* $\{W_t | t \geq 0\}$ *be a standard Brownian motion. For each* $a < 0$, $b > 0$, *and* $t > 0$, *define sets as*

$$A \doteq \left\{\tau_a^W < \tau_b^W\right\}, \quad A_t \doteq \left\{\tau_a^W < t\right\}, \quad B_t \doteq \left\{\tau_b^W < t\right\}.$$

For a fixed y *and a Borel set* H, *define functions as*

$$R_y(z) \doteq 2y - z, \qquad R_y(H) \doteq \{R_y(h) \mid h \in H\},$$
$$I^W(H) \doteq \{\omega \mid W_t(\omega) \in H\}, \quad I_y^W(H) \doteq \{\omega \mid W_t(\omega) \in R_y(H)\}.$$

Then, the following equalities hold;

$$P\left(A^c \cap I^W(H)\right) = P\left(I_b^W(H)\right) - P\left(A \cap I_b^W(H)\right), \quad (H \subset (-\infty, a]),$$

$$P\left(A \cap I^W(H)\right) = P\left(I_a^W(H)\right) - P\left(A^c \cap I_a^W(H)\right), \quad (H \subset [b, \infty)). \qquad \square$$

From now on, we assume a and b are constants satisfying

$$a < 0 < b. \tag{2.3}$$

Using Lemmas 1 and 2, we can derive the joint probability of $W_t \in I$ and $a \leq m_t^W \leq M_t^W \leq b$.

Lemma 3. *The joint probability of $W_t \in I$ and $a \le m_t^W \le M_t^W \le b$ is*

$$\Pr\left(a \le m_t^W \le M_t^W \le b, W_t \in I\right) = \int_I p_G^W(a, b, w; t)\, dw,$$

where the trivariate joint probability density $p_G^W(a, b, w; t)$ is

$$p_G^W(a, b, w; t) \doteq \sum_{k=-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t} \{w - 2k[b-a]\}^2 \right) \right.$$
$$\left. - \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t} \{w - 2b + 2k[b-a]\}^2 \right) \right].$$

□

This lemma and its variants are found in the probabilistic literature such as Bachelier (1901, pp. 192-194, 1941), Lévy (1948, p. 213), Doob (1949), Darling and Siegert (1953), Cox and Miller (1965, p. 222), Billingsley (1968, pp. 77-79), Freedman (1970, pp. 26-7), Feller (1971, p. 341), Borodin and Salminen (2002, p. 174), etc.

2.2 Fourier Series Solution

It is well-known (see, *e.g.*, Evans [1998, p. 46]) that a normal probability density function $\phi(w\,;\, m, t)$ satisfies the Kolmogorov-Fokker-Planck equation. More specifically, $p_G^W(a, b, w; t)$ satisfies

$$\frac{\partial p(a, b, w; t)}{\partial t} = \frac{1}{2} \frac{\partial^2 p(a, b, w; t)}{\partial w^2}. \tag{2.4}$$

It is known (see, *e.g.*, Cox and Miller [1965, pp. 222-3]) that $p_G^W(a, b, w; t)$ satisfies the initial and boundary conditions

$$p(a, b, w; 0) = \delta(w), \tag{2.5}$$

$$p(a, b, a; t) = 0, \quad (t > 0), \tag{2.6}$$

$$p(a, b, b; t) = 0, \quad (t > 0). \tag{2.7}$$

Thus, we may call $p_G^W(a, b, x; t)$ the Gaussian series solution of the boundary value problem composed of the PDE (2.4), the initial condition (2.5), and the boundary conditions (2.6) and (2.7). We can solve the boundary value problem using separation of variables, and obtain the Fourier series solution. See *e.g.*, Cox and Miller (1965, pp. 222-223).

Lemma 4. *The joint probability of* $W_t \in I$ *and* $a \leq m_t^W \leq M_t^W \leq b$ *is*

$$\Pr\left(a \leq m_t^W \leq M_t^W \leq b, W_t \in I\right) = \int_I p_G^W(a, b, w; t) dw,$$

where the trivariate joint probability density $p_G^W(a, b, w; t)$ *is*

$$p_F^W(a, b, w; t) \doteq \sum_{n=1}^{\infty} \frac{2}{b-a} \exp\left(-\frac{1}{2}\frac{\pi^2 n^2 t}{[b-a]^2}\right) \sin\left(\frac{-a\pi n}{b-a}\right) \sin\left(\frac{\pi n\,[w-a]}{b-a}\right).$$

□

Since $p_G^W(a, b, w; t)$ and $p_F^W(a, b, w; t)$ are the solutions of the same boundary value problem, they should be the same. We can prove the equivalence through several methods. Among them, one is using the Shah function Ш. See, *e.g.*, Bracewell (2000, p. 82). Another is using Poisson summation formula. See, *e.g.*, Zwillinger (2003, p. 48).

2.3 Gaussian Product Solution

To present the Gaussian series solution $p_G^W(a, b, w; t)$ as a difference of two infinite products, we define three functions as

$$\alpha \doteq \exp\left(-\frac{4[b-a]^2}{t}\right), \tag{2.8}$$

$$\beta(w) \doteq \exp\left(\frac{2[b-a][w-b+a]}{t}\right), \tag{2.9}$$

$$\gamma(w) \doteq \beta(w) \exp\left(-\frac{4b[b-a]}{t}\right). \tag{2.10}$$

Jacobi's triple product identity is

$$\sum_{k=-\infty}^{\infty} A^{k[k-1]/2} B^k = \prod_{j=1}^{\infty} \left\{ \left[1 - A^j\right] \left[1 + \frac{1}{B} A^j\right] \left[1 + BA^{j-1}\right] \right\} \tag{2.11}$$

for any complex numbers A satisfying $|A| < 1$ and B. For Eq. (2.11), readers may refer to Zwillinger (2003, p. 48). Using Eq. (2.11), we get

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t} \{w - 2k[b-a]\}^2\right) dw \\
&= \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t} w^2\right) \sum_{k=-\infty}^{\infty} \beta^k(w) \alpha^{k[k-1]/2} \\
&= \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t} w^2\right) \prod_{j=1}^{\infty} \left\{ \left[1 - \alpha^j\right] \left[1 + 2\cosh\left(\frac{2[b-a]w}{t}\right) \alpha^{j-1/2} + \alpha^{2j-1}\right] \right\},
\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t} \{w - 2b - 2k[b-a]\}^2\right) dw \\
&= \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t} [w - 2b]^2\right) \sum_{k=-\infty}^{\infty} \gamma^k(w) \alpha^{k[k-1]/2} \\
&= \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t} [w - 2b]^2\right) \\
&\quad \cdot \prod_{j=1}^{\infty} \left\{ \left[1 - \alpha^j\right] \left[1 + 2\cosh\left(\frac{2[b-a][w-2b]}{t}\right) \alpha^{j-1/2} + \alpha^{2j-1}\right] \right\}.
\end{aligned} \tag{2.13}$$

Substituting Eqs. (2.12) and (2.13) into Lemma 3, we get the following proposition.

Proposition 1. *For $a < 0 < b$, the trivariate joint probability density $p_G^W(a, b, w; t)$*

is equal to

$$
\begin{aligned}
& p_{GP}^{W}(a,b,w;t) \\
\doteq\ & \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}w^2\right) \prod_{j=1}^{\infty} \left\{ \left[1-\alpha^j\right] \left[1+2\cosh\left(\frac{2[b-a]w}{t}\right)\alpha^{j-1/2}+\alpha^{2j-1}\right] \right\} \\
& - \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}[w-2b]^2\right) \\
& \quad \cdot \prod_{j=1}^{\infty} \left\{ \left[1-\alpha^j\right] \left[1+2\cosh\left(\frac{2[b-a][w-2b]}{t}\right)\alpha^{j-1/2}+\alpha^{2j-1}\right] \right\},
\end{aligned}
$$

where $\alpha = \exp\left(-4[b-a]^2/t\right)$. □

The infinite product $p_{GP}^{W}(a,b,w;t)$ is different from the one presented by Choi and Roh (2013).

2.4 Fourier Product Solution

To present the Fourier series solution $p_F^W(a,b,w;t)$ as a difference of two infinite products, we define three functions as

$$\theta \doteq \exp\left(-\frac{\pi^2 t}{[b-a]^2}\right), \tag{2.14}$$

$$\eta(w) \doteq \exp\left(i\frac{\pi w}{b-a} - \frac{\pi^2 t}{2[b-a]^2}\right), \tag{2.15}$$

$$\zeta(w) \doteq \exp\left(i\frac{\pi[2a-w]}{b-a} - \frac{\pi^2 t}{2[b-a]^2}\right). \tag{2.16}$$

We know from Lemma 4 that

$$
\begin{aligned}
p_F^W(a,b,w;t) = & \frac{1}{2[b-a]} \sum_{k=-\infty}^{\infty} \exp\left(i\frac{\pi k w}{b-a}\right) \exp\left(-\frac{\pi^2 k^2 t}{2[b-a]^2}\right) \\
& - \frac{1}{2[b-a]} \sum_{k=-\infty}^{\infty} \exp\left(i\frac{\pi k[w-2a]}{b-a}\right) \exp\left(-\frac{\pi^2 k^2 t}{2[b-a]^2}\right).
\end{aligned}
\tag{2.17}
$$

It is clear that

$$
\begin{aligned}
&\frac{1}{2[b-a]}\sum_{k=-\infty}^{\infty}\exp\left(i\frac{\pi k w}{b-a}\right)\exp\left(-\frac{\pi^2k^2t}{2[b-a]^2}\right)\\
&=\frac{1}{2[b-a]}\sum_{k=-\infty}^{\infty}\exp\left(i\frac{\pi k w}{b-a}-\frac{\pi^2 kt}{2[b-a]^2}\right)\exp\left(-\frac{\pi^2tk[k-1]}{2[b-a]^2}\right). \qquad (2.18)
\end{aligned}
$$

Applying Eq. (2.11) to Eq. (2.18) yields

$$
\begin{aligned}
&\frac{1}{2[b-a]}\sum_{k=-\infty}^{\infty}\exp\left(i\frac{\pi k w}{b-a}\right)\exp\left(-\frac{\pi^2k^2t}{2[b-a]^2}\right)\\
&=\frac{1}{2[b-a]}\prod_{j=1}^{\infty}\left\{\left[1-\theta^j\right]\left[1+\frac{1}{\eta(w)}\theta^j\right]\left[1+\eta(w)\theta^{j-1}\right]\right\}\\
&=\frac{1}{2[b-a]}\prod_{j=1}^{\infty}\left\{\left[1-\theta^j\right]\left[1+2\cos\left(\frac{\pi w}{b-a}\right)\theta^{j-1/2}+\theta^{2j-1}\right]\right\}. \qquad (2.19)
\end{aligned}
$$

Using a similar method, we can show that

$$
\begin{aligned}
&\frac{1}{2[b-a]}\sum_{k=-\infty}^{\infty}\exp\left(i\frac{\pi k [w-2a]}{b-a}\right)\exp\left(-\frac{\pi^2k^2t}{2[b-a]^2}\right)\\
&=\frac{1}{2[b-a]}\prod_{j=1}^{\infty}\left\{\left[1-\theta^j\right]\left[1+\frac{1}{\zeta(w)}\theta^j\right]\left[1+\zeta(w)\theta^{j-1}\right]\right\}\\
&=\frac{1}{2[b-a]}\prod_{j=1}^{\infty}\left\{\left[1-\theta^j\right]\left[1+2\cos\left(\frac{\pi [2a-w]}{b-a}\right)\theta^{j-1/2}+\theta^{2j-1}\right]\right\}. \qquad (2.20)
\end{aligned}
$$

From Eqs. (2.17), (2.19), and (2.20) we get the following proposition.

Proposition 2. *For $a<0<b$, the trivariate joint probability density $p_F^W(a,b,w;t)$ is equal to*

$$
\begin{aligned}
&p_{FP}^W(a,b,w;t)\\
&\doteq\frac{1}{2[b-a]}\prod_{j=1}^{\infty}\left\{\left[1-\theta^j\right]\left[1+2\cos\left(\frac{\pi w}{b-a}\right)\theta^{j-1/2}+\theta^{2j-1}\right]\right\}\\
&\quad-\frac{1}{2[b-a]}\prod_{j=1}^{\infty}\left\{\left[1-\theta^j\right]\left[1+2\cos\left(\frac{\pi [2a-w]}{b-a}\right)\theta^{j-1/2}+\theta^{2j-1}\right]\right\},
\end{aligned}
$$

where $\theta=\exp\left(-\pi^2t/[b-a]^2\right)$. □

2.5 A General Form Of The Trivariate Joint Probability

The trivariate joint probability density of a Brownian motion and its maximum and minimum is expressed by the Gaussian series solution $p_G^W(a, b, w; t)$, the Fourier series solution $p_F^W(a, b, w; t)$, the Gaussian product solution $p_{GP}^W(a, b, w; t)$, and the Fourier product solution $p_{FP}^W(a, b, w; t)$. Thus, we can present a general form of the trivariate joint probability density as follows.

Theorem 1. *Consider a standard Brownian motion* $\{W_t | t \geq 0\}$ *with* $W_0 = 0$ *and its maximum* M_t^W *and minimum* m_t^W. *Then,* $p^W(a, b, w; t)$ *defined by*

$$\begin{aligned} p^W(a, b, w; t) \doteq\ & c_1 p_G^W(a, b, w; t) + c_2 p_F^W(a, b, w; t) \\ & + c_3 p_{GP}^W(a, b, w; t) + c_4 p_{FP}^W(a, b, w; t) \end{aligned}$$

with nonnegative constants $c_1, c_2, c_3,\ c_4$ *satisfying* $\sum_{i=1}^{4} c_i = 1$ *is the trivariate joint probability density of* $\left(W_t, m_t^W, M_t^W\right)$. □

Lemma 3 implies the probability that the Brownian motion starting from $W_0 = 0$ reaches neither the upper barrier b nor the lower barrier a before time t is

$$\begin{aligned} & P\left(a \leq m_t^W \leq M_t^W \leq b\right) = \int_a^b p_G^W(a, b, w; t) dw \\ = & \sum_{k=-\infty}^{\infty} \left[N\left(\frac{b - 2k[b-a]}{\sqrt{t}}\right) - N\left(\frac{a - 2k[b-a]}{\sqrt{t}}\right)\right] \\ & - \sum_{k=-\infty}^{\infty} \left[N\left(\frac{-b - 2k[b-a]}{\sqrt{t}}\right) - N\left(\frac{a - 2b - 2k[b-a]}{\sqrt{t}}\right)\right], \end{aligned} \tag{2.21}$$

where $N(\cdot)$ is the cumulative distribution of a standard normal random variable. Feller (1971, p. 342) calls this the total probability mass at epoch t. Also, Lemma 4 implies the total probability mass at epoch t is

$$P\left(a \le m_t^W \le M_t^W \le b\right) = \int_a^b p_F^W(a, b, w; t) dw$$
$$= \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{1}{2m-1} \sin\left(\frac{-\pi[2m-1]a}{b-a}\right) \exp\left(-\frac{\pi^2[2m-1]^2 t}{2[b-a]^2}\right). \tag{2.22}$$

For each step of infinite summation $P\left(a \le m_t^W \le M_t^W \le b\right)$ in Eq. (2.21), we need to calculate four integrals of the normal probability density function that does not have an explicit anti-derivative. However, we do not need any integration when we use Eq. (2.22). Thus, it is computationally more efficient to use $p_F^W(a, b, w; t)$ than $p_G^W(a, b, w; t)$ to calculate$P\left(a \le m_t^W \le M_t^W \le b\right)$.

When we calculate $p_{GP}^W(a, b, w; t)$, we first do α, $\cosh\left(\frac{2[b-a]w}{t}\right)$, and $\cosh\left(\frac{2[b-a][w-2b]}{t}\right)$, and then we need to do neither an exponential function except α^j and $\alpha^{j-1/2}$ nor a trigonometric function in the j-step of the infinite production. Similarly, when we calculate $p_{FP}^W(a, b, w; t)$, we first do θ, $\cos\left(\frac{\pi w}{[b-a]}\right)$, and $\cos\left(\frac{\pi[2a-w]}{b-a}\right)$, and then we need to do neither an exponential function except θ^j and $\theta^{j-1/2}$ nor a trigonometric function in the j-step of the infinite production. Moreover, we know an addition and a multiplication have the same flop, or computational complexity. Thus, the product formulas $p_{GP}^W(a, b, w; t)$ and $p_{FP}^W(a, b, w; t)$ are computationally more efficient than the series formulas $p_G^W(a, b, w; t)$ and $p_F^W(a, b, w; t)$.

3 Trivariate joint probability density of Generalized Brownian motion

Assume $\{X_u \mid u \ge 0\}$ satisfies

$$dX_u = \nu du + \sigma dW_u, \tag{3.1}$$

where $\{W_u \,|\, u \geq 0\}$ is a standard Brownian motion with $W_0 = 0$. Then, $\{X_u\}$ is a Brownian motion with drift parameter ν and instantaneous variance σ^2. Denote its starting value by x_0. Eq. (3.1) implies

$$X_t = x_0 + \nu t + \sigma W_t. \tag{3.2}$$

Denote the maximum and the minimum of $\{X_u \,|\, 0 \leq u \leq t\}$, respectively, by

$$M_t^X \doteq \max_{0 \leq s \leq t} X_s \quad \text{and} \quad m_t^X \doteq \min_{0 \leq s \leq t} X_s. \tag{3.3}$$

Girsanov's theorem is especially important in pricing derivatives. A simple version of the theorem is as follows.

Lemma 5. *Consider a standard Brownian motion $\{W_t \,|\, 0 \leq t \leq T\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\{\mathcal{F}_t \,|\, 0 \leq t \leq T\}$ satisfying $\mathcal{F}_T = \mathcal{F}$. For an $\{\mathcal{F}_t\}$-adapted process $\{\theta_t \,|\, 0 \leq t \leq T\}$, we set*

$$\begin{aligned}
\xi_t &\doteq \exp\left(-\int_0^t \theta_u dW_u - \frac{1}{2}\int_0^t \theta_u^2 du\right), \quad (0 \leq t \leq T), \\
W_t^Q &\doteq W_t + \int_0^t \theta_u du, \quad (0 \leq t \leq T), \\
Q(A) &\doteq \int_A \xi_T(\omega) dP(\omega), \quad (A \in \mathcal{F}).
\end{aligned}$$

Assume that $E\left(\int_0^T \theta_u^2 \xi_u^2 du\right) < \infty$. Then, Q is a probability measure, and $\left\{W_t^Q \,|\, 0 \leq t \leq T\right\}$ is a standard Brownian motion under the probability measure Q. □

Lemma 5 is due to Cameron and Martin (1944), Maruyama (1954, 1955), and Girsanov (1960). The expectation of a random variable X with respect to the probability measure Q will be denoted as $E^Q(X)$. The following lemma can be easily proved.

Lemma 6. *Let α and β be constants. Then,*

$$\exp(\alpha x+\beta)\phi\left(x;m,v\right)=\exp\left(\alpha m+\beta+\frac{1}{2}\alpha^2 v\right)\phi\left(x;m+\alpha v,v\right).$$

$\square$

3.1 Gaussian Series Solution

Let us derive the Gaussian series solution of the joint probability of $X_t \in I^X$ and $a^X \leq m_t^X \leq M_t^X \leq b^X$, where a^X and b^X are constants satisfying

$$a^X < x_0 + \nu t < b^X. \tag{3.4}$$

Eq. (3.4) corresponds to Eq. (2.3).

We first consider the case $\nu = 0$ and $x_0 = 0$. Set $\tilde{X}_t \doteq X_t/\sigma$. Then, we know

$$\begin{aligned}
&\Pr\left(X_t \in I^X, a^X \leq m_t^X \leq M_t^X \leq b^X\right)\\
&=\Pr\left(\tilde{X}_t \in \frac{1}{\sigma}I^X, \frac{1}{\sigma}a^X \leq m_t^{\tilde{X}} \leq M_t^{\tilde{X}} \leq \frac{1}{\sigma}b^X\right)\\
&=\sum_{k=-\infty}^{\infty}\int_{I^X/\sigma}\phi\left(\tilde{x};\frac{2k\left[b^X-a^X\right]}{\sigma},t\right)d\tilde{x}\\
&\quad-\sum_{k=-\infty}^{\infty}\int_{I^X/\sigma}\phi\left(\tilde{x};\frac{2b^X-2k\left[b^X-a^X\right]}{\sigma},t\right)d\tilde{x}\\
&=\sum_{k=-\infty}^{\infty}\int_{I^X}\phi\left(x;2k\left[b^X-a^X\right],\sigma^2 t\right)dx\\
&\quad-\sum_{k=-\infty}^{\infty}\int_{I^X}\phi\left(x;2b^X-2k\left[b^X-a^X\right],\sigma^2 t\right)dx,
\end{aligned} \tag{3.5}$$

where the second equality holds by Lemma 3, and the third one does by the transform $x = \sigma\tilde{x}$.

Secondly, we consider the case $\nu \neq 0$ and $x_0 = 0$. Eq. (3.2) becomes

$$X_t = \nu t + \sigma W_t. \tag{3.6}$$

It is clear that

$$X_t \overset{d}{\sim} N\left(\nu t, \sigma^2 t\right). \tag{3.7}$$

Setting $\theta_u = \nu/\sigma$ in Lemma 5, we get

$$\xi_t = \exp\left(-\frac{\nu}{\sigma} W_t - \frac{\nu^2}{2\sigma^2} t\right). \tag{3.8}$$

Eqs. (3.6) and (3.8) imply

$$\xi_t = \exp\left(-\frac{\nu}{\sigma^2} X_t + \frac{\nu^2}{2\sigma^2} t\right). \tag{3.9}$$

Thus,

$$\xi_t^{-1} = \exp\left(\frac{\nu}{\sigma^2} X_t - \frac{\nu^2}{2\sigma^2} t\right). \tag{3.10}$$

Let Q be the corresponding equivalent martingale measure. Then,

$$\begin{aligned}
&\Pr\left(X_t \in I^X, a^X \leq m_t^X \leq M_t^X \leq b^X\right) \\
&= E\left(1\left(X_t \in I^X, a^X \leq m_t^X \leq M_t^X \leq b^X\right)\right) \\
&= E^Q\left(1\left(X_t \in I^X, a^X \leq m_t^X \leq M_t^X \leq b^X\right) \xi_t^{-1}\right) \\
&= E^Q\left(1\left(X_t \in I^X, a^X \leq m_t^X \leq M_t^X \leq b^X\right) \exp\left(\frac{\nu}{\sigma^2} X_t - \frac{\nu^2}{2\sigma^2} t\right)\right),
\end{aligned} \tag{3.11}$$

where the second equality holds by Lemma 5, and the third equality does by Eq. (3.10). Since $\{X_u/\sigma \,|\, 0 \leq u \leq t\}$ is a standard Brownian motion with under the probability measure Q, Eq. (3.5) implies

$$\begin{aligned}
&E^Q\left(\exp\left(\frac{\nu}{\sigma^2} X_t - \frac{\nu^2}{2\sigma^2} t\right) 1\left(X_t \in I^X, a^X \leq m_t^X \leq M_t^X \leq b^X\right)\right) \\
&= \sum_{k=-\infty}^{\infty} \int_{I^X} \exp\left(\frac{\nu}{\sigma^2} x - \frac{\nu^2}{2\sigma^2} t\right) \phi\left(x; 2k\left[b^X - a^X\right], \sigma^2 t\right) dx \\
&\quad - \sum_{k=-\infty}^{\infty} \int_{I^X} \exp\left(\frac{\nu}{\sigma^2} x - \frac{\nu^2}{2\sigma^2} t\right) \phi\left(x; 2b^X - 2k\left[b^X - a^X\right], \sigma^2 t\right) dx.
\end{aligned} \tag{3.12}$$

Using Lemma 6, we can show that

$$
\begin{aligned}
& \exp\left(\frac{\nu}{\sigma^2}x - \frac{\nu^2}{2\sigma^2}t\right)\phi\left(x; 2k\left[b^X - a^X\right], \sigma^2 t\right) \\
= \ & \exp\left(\frac{2k\left[b^X - a^X\right]\nu}{\sigma^2}\right)\phi\left(x; \nu t + 2k\left[b^X - a^X\right], \sigma^2 t\right), \qquad (3.13) \\
& \exp\left(\frac{\nu}{\sigma^2}x - \frac{\nu^2}{2\sigma^2}t\right)\phi\left(x; 2b^X - 2k\left[b^X - a^X\right], \sigma^2 t\right) dx \\
= \ & \exp\left(\frac{2\left\{b^X - k\left[b^X - a^X\right]\right\}\nu}{\sigma^2}\right)\phi\left(x; \nu t + 2\left\{b^X - k\left[b^X - a^X\right]\right\}, \sigma^2 t\right). \qquad (3.14)
\end{aligned}
$$

We know from Eqs. (3.11) $\sim$ (3.14) that the joint probability of $X_t \in I^X$ and $a^X \le m_t^X \le M_t^X \le b^X$ is

$$
\Pr\left(a^X \le m_t^X \le M_t^X \le b^X, X_t \in I^X\right) = \int_{I^X} p_G^{X,0}\left(a^X, b^X, x; t\right) dx, \qquad (3.15)
$$

where the trivariate joint probability density $p_G^{X,0}\left(a^X, b^X, x; t\right)$ is

$$
\begin{aligned}
& p_G^{X,0}\left(a^X, b^X, x; t\right) \\
\doteq & \sum_{k=-\infty}^{\infty} \exp\left(\frac{2k\left[b^X - a^X\right]\nu}{\sigma^2}\right)\phi\left(x; \nu t + 2k\left[b^X - a^X\right], \sigma^2 t\right) \\
& - \sum_{k=-\infty}^{\infty} \exp\left(\frac{2\left\{ka^X - [k-1]b^X\right\}\nu}{\sigma^2}\right)\phi\left(x; \nu t + 2\left\{ka^X - [k-1]b^X\right\}, \sigma^2 t\right). \qquad (3.16)
\end{aligned}
$$

Thirdly, we consider the case $\nu \neq 0$ and $x_0 \neq 0$. Substituting $\widehat{X}_t = X_t - x_0$, $a^{\widehat{X}} = a^X - x_0$, and $b^{\widehat{X}} = b^X - x_0$ into Eq. (3.16), we know the trivariate joint probability density $p_G^X\left(a^X, b^X, x; t\right)$ is

$$
\begin{aligned}
& p_G^X\left(a^X, b^X, x; t\right) \\
&= p_G^{\widehat{X},0}\left(a^{\widehat{X}}, b^{\widehat{X}}, x - x_0; t\right) \\
&= \sum_{k=-\infty}^{\infty} \exp\left(\frac{2k\left[b^{\widehat{X}} - a^{\widehat{X}}\right]\nu}{\sigma^2}\right)\phi\left(x - x_0; \nu t + 2k\left[b^{\widehat{X}} - a^{\widehat{X}}\right], \sigma^2 t\right) \\
&\quad - \sum_{k=-\infty}^{\infty} \exp\left(\frac{2\left\{ka^{\widehat{X}} - [k-1]b^{\widehat{X}} - x_0\right\}\nu}{\sigma^2}\right) \\
&\qquad \cdot \phi\left(x - x_0; \nu t + 2\left\{ka^{\widehat{X}} - [k-1]b^{\widehat{X}}\right\}, \sigma^2 t\right).
\end{aligned} \tag{3.17}
$$

Thus, we get the following lemma.

Lemma 7. *The joint probability of $X_t \in I^X$ and $a \leq m_t^X \leq M_t^X \leq b$ of the generalized Brown motion is*

$$
\Pr\left(a \leq m_t^X \leq M_t^X \leq b, X_t \in I^X\right) = \int_{I^X} p_G^X(a, b, x; t)\, dx,
$$

where the trivariate joint probability density $p_G^X(a, b, x; t)$ is

$$
\begin{aligned}
& p_G^X\left(a^X, b^X, x; t\right) \\
&\doteq \sum_{k=-\infty}^{\infty} \exp\left(\frac{2k\left[b^X - a^X\right]\nu}{\sigma^2}\right)\phi\left(x; x_0 + \nu t + 2k\left[b^X - a^X\right], \sigma^2 t\right) \\
&\quad - \sum_{k=-\infty}^{\infty} \exp\left(\frac{2\left\{ka^X - [k-1]b^X - x_0\right\}\nu}{\sigma^2}\right) \\
&\qquad \cdot \phi\left(x; x_0 + \nu t + 2\left\{ka^X - [k-1]b^X\right\}, \sigma^2 t\right).
\end{aligned}
$$

$\square$

3.2 Fourier Series Solution

Let us derive the Fourier series solution of the joint probability of $X_t \in I^X$ and $a^X \leq m_t^X \leq M_t^X \leq b^X$, where a^X and b^X satisfy Eq. (3.4).

We first consider the case $\nu = 0$ and $x_0 = 0$. Set $\tilde{X}_t \doteq X_t/\sigma$. We know

$$\begin{aligned}
&\Pr\left(X_t \in I^X, a^X \leq m_t^X \leq M_t^X \leq b^X\right) \\
&= \Pr\left(\tilde{X}_t \in \frac{1}{\sigma} I^X, \frac{1}{\sigma} a^X \leq m_t^{\tilde{X}} \leq M_t^{\tilde{X}} \leq \frac{1}{\sigma} b^X\right) \\
&= \sum_{n=1}^{\infty} \int_{I^X/\sigma} \frac{2\sigma}{b^X - a^X} \exp\left(-\frac{1}{2} \frac{\sigma^2 \pi^2 n^2 t}{\left[b^X - a^X\right]^2}\right) \\
&\qquad \cdot \sin\left(\frac{-a^X \pi n}{b^X - a^X}\right) \sin\left(\frac{\pi n \left[\sigma \tilde{x} - a^X\right]}{b^X - a^X}\right) d\tilde{x} \\
&= \sum_{n=1}^{\infty} \int_{I^X} \frac{2}{b^X - a^X} \exp\left(-\frac{1}{2} \frac{\sigma^2 \pi^2 n^2 t}{\left[b^X - a^X\right]^2}\right) \\
&\qquad \cdot \sin\left(\frac{-a^X \pi n}{b^X - a^X}\right) \sin\left(\frac{\pi n \left[x - a^X\right]}{b^X - a^X}\right) dx,
\end{aligned} \tag{3.18}$$

where the second equality holds by Lemma 4, and the third one does by the transform $x = \sigma \tilde{x}$.

Secondly, we consider the case $\nu \neq 0$ and $x_0 = 0$. Setting $\theta_u = \nu/\sigma$ in Lemma 5, we know from Eq. (3.11) that

$$\begin{aligned}
&\Pr\left(X_t \in I^X, a^X \leq m_t^X \leq M_t^X \leq b^X\right) \\
&= E^Q\left(1\left(X_t \in I^X, a^X \leq m_t^X \leq M_t^X \leq b^X\right) \exp\left(\frac{\nu}{\sigma^2} X_t - \frac{\nu^2}{2\sigma^2} t\right)\right).
\end{aligned} \tag{3.19}$$

Since $\{X_u/\sigma \,|\, 0 \leq u \leq t\}$ is a standard Brownian motion with under the probability measure Q, Eq. (3.18) implies

$$\begin{aligned}
&E^Q\left(\exp\left(\frac{\nu}{\sigma^2} X_t - \frac{\nu^2}{2\sigma^2} t\right) 1\left(X_t \in I^X, a^X \leq m_t^X \leq M_t^X \leq b^X\right)\right) \\
&= \sum_{n=1}^{\infty} \int_{I^X} \exp\left(\frac{\nu}{\sigma^2} x - \frac{\nu^2}{2\sigma^2} t\right) \frac{2}{b^X - a^X} \exp\left(-\frac{1}{2} \frac{\sigma^2 \pi^2 n^2 t}{\left[b^X - a^X\right]^2}\right) \\
&\qquad \cdot \sin\left(\frac{-a^X \pi n}{b^X - a^X}\right) \sin\left(\frac{\pi n \left[x - a^X\right]}{b^X - a^X}\right) dx.
\end{aligned} \tag{3.20}$$

Thus, the joint probability of $X_t \in I^X$ and $a^X \leq m_t^X \leq M_t^X \leq b^X$ is

$$\Pr\left(a^X \leq m_t^X \leq M_t^X \leq b^X, X_t \in I^X\right) = \int_{I^X} p_F^{X,0}\left(a^X, b^X, x; t\right) dx, \tag{3.21}$$

where the trivariate joint probability density $p_F^{X,0}\left(a^X, b^X, x; t\right)$ is

$$\begin{aligned} & p_F^{X,0}\left(a^X, b^X, x; t\right) \\ = & \exp\left(\frac{\nu x}{\sigma^2}\right) \sum_{n=1}^{\infty} \frac{2}{b^X - a^X} \exp\left(-\frac{1}{2}\left\{\frac{\nu^2}{\sigma^2} + \frac{\pi^2\sigma^2 n^2}{\left[b^X - a^X\right]^2}\right\} t\right) \\ & \cdot \sin\left(\frac{-a^X \pi n}{b^X - a^X}\right) \sin\left(\frac{\pi n\left[x - a^X\right]}{b^X - a^X}\right). \end{aligned} \tag{3.22}$$

Thirdly, we consider the case $\nu \neq 0$ and $x_0 \neq 0$. Substituting $\widehat{X}_t = X_t - x_0$, $a^{\widehat{X}} = a^X - x_0$, and $b^{\widehat{X}} = b^X - x_0$ into Eq. (3.22), we get the following lemma.

Lemma 8. *The joint probability of $X_t \in I$ and $a \leq m_t^X \leq M_t^X \leq b$ is*

$$\Pr\left(a \leq m_t^X \leq M_t^X \leq b, X_t \in I\right) = \int_I p_F^X(a, b, x; t) dx,$$

where the trivariate joint probability density $p_F^X(a, b, x; t)$ is

$$\begin{aligned} & p_F^X\left(a^X, b^X, x; t\right) \\ = & \exp\left(\frac{\nu\left[x - x_0\right]}{\sigma^2}\right) \sum_{n=1}^{\infty} \frac{2}{b^X - a^X} \exp\left(-\frac{1}{2}\left\{\frac{\nu^2}{\sigma^2} + \frac{\pi^2\sigma^2 n^2}{\left[b^X - a^X\right]^2}\right\} t\right) \\ & \cdot \sin\left(\frac{\pi n\left[x_0 - a^X\right]}{b^X - a^X}\right) \sin\left(\frac{\pi n\left[x - a^X\right]}{b^X - a^X}\right). \end{aligned}$$

□

It is well-known (see, *e.g.*, Cox and Miller [1967, p. 215] and Choi and Roh [2013]) that the transition probability density function $p\left(x_s, s; x_u, u\right)$ for $s < u$ of the diffusion process in Eq. (3.1) satisfies the Kolmogorov-Fokker-Planck equation

$$\frac{\partial p\left(x_s, s; x_u, u\right)}{\partial u} = \frac{1}{2}\sigma^2 \frac{\partial^2 p\left(x_s, s; x_u, u\right)}{\partial x_u^2} - \nu \frac{\partial p\left(x_s, s; x_u, u\right)}{\partial x_u}. \tag{3.23}$$

Suppose there exist two absorbing barriers a^X and b^X. Then, the initial and boundary conditions become

$$p\left(x_0, 0; x, 0\right) = \delta\left(x - x_0\right), \tag{3.24}$$

$$p\left(x_0, 0; x_u, a^X\right) = 0, \quad (u > 0), \tag{3.25}$$

$$p\left(x_0, 0; x_u, b^X\right) = 0, \quad (u > 0), \tag{3.26}$$

where $\delta(x)$ is the Dirac delta function, which is a distribution defined by $\delta(x) = 0$ for $x \neq 0$); otherwise, ∞ and is normalized so that $\int_{-\infty}^{\infty} \delta(x)dx = 1$. As Cox and Miller (1967, p. 222) state, we place sources at the points $2k\left[b^X - a^X\right]$ with strengths $\exp\left(2k\left[b^X - a^X\right]\nu/\sigma^2\right)$ and sources at the points $2ka^X - 2[k-1]b^X$ with strengths $-\exp\left(\left\{2ka^X - 2[k-1]b^X\right\}\nu/\sigma^2\right)$. Then, the linear superposition of solutions for each source weighted by the corresponding strength becomes the solution, *i.e.*,

$$\begin{aligned} p\left(x_0, 0; x, t\right) \quad = \quad & \sum_{k=-\infty}^{\infty} \exp\left(2k\left[b^X - a^X\right]\frac{\nu}{\sigma^2}\right) \\ & \qquad \cdot\phi\left(x; x_0 + \nu t + 2k\left[b^X - a^X\right], \sigma^2 t\right) \\ & - \sum_{k=-\infty}^{\infty} \exp\left(2\left\{ka^X - [k-1]b^X - x_0\right\}\frac{\nu}{\sigma^2}\right) \\ & \qquad \cdot\phi\left(x; x_0 + \nu t + \left\{2ka^X - 2[k-1]b^X\right\}, \sigma^2 t\right), \end{aligned} \tag{3.27}$$

which equals to the Gaussian series solution in Lemma 7. Solving the boundary value problem consisting of the PDE (3.23), the initial condition (3.24), and the boundary conditions (3.25) and (3.26) using separation of variables, we get

$$\begin{aligned} & p\left(x_0, 0; x, t\right) \\ & \doteq \exp\left(\frac{\nu\left[x - x_0\right]}{\sigma^2}\right) \sum_{k=1}^{\infty} \exp\left(-\frac{1}{2}\left\{\left(\frac{\nu}{\sigma}\right)^2 + \left(\frac{-\pi a^X k}{b^X - a^X}\right)^2\right\} t\right) \\ & \qquad \cdot \frac{2}{b^X - a^X} \sin\frac{\pi a^X k}{b^X - a^X} \sin\frac{\pi\left[x - a^X\right]k}{b^X - a^X}, \end{aligned} \tag{3.28}$$

which equals to the Fourier series solution in Lemma 8.

4 Trivariate joint probability density of Geometric Brownian motion

Set $S_u \doteq \exp(X_u)$, $s_0 \doteq \exp(x_0)$, and $\mu \doteq \nu + \sigma^2/2$, where $\{X_u \,|\, u \geq 0\}$ is the geometric Brownian motion satisfying Eq. (3.2). Ito-Doeblin lemma implies $\{S_u\}$ satisfies the stochastic differential equation;

$$dS_u = \mu S_u du + \sigma S_u dW_u, \tag{4.1}$$

where $\{W_u \mid u \geq 0\}$ is a standard Brownian motion with $W_0 = 0$. We call μ and σ the drift and the volatility, respectively. Denote its maximum and minimum by

$$M_t^S \doteq \max_{0 \leq u \leq t} S_u \quad \text{and} \quad m_t^S \doteq \min_{0 \leq u \leq t} S_u. \tag{4.2}$$

Let a^S and b^S be constants satisfying

$$a^S < s_0 \exp(\nu t) < b^S. \tag{4.3}$$

Eq. (4.3) corresponds to Eq. (3.4). Also, assume that $I^S \subset \left[a^S, b^S\right]$ is a Borel set.

4.1 Gaussian Series Solution

Applying $X_u = \ln S_u$ to Eq. (7), we know the trivariate joint probability density of $S_t \in I^S$ and $a^S \leq m_t^S \leq M_t^S \leq b^S$ is

$$\begin{aligned}
& p_G^S\left(a^S, b^S, s; t\right) \\
& \doteq \sum_{k=-\infty}^{\infty} \exp\left(\frac{2k\left[\ln b^S - \ln a^S\right]\nu}{\sigma^2}\right) \phi\left(\ln s; \ln s_0 + \nu t + 2k\left[\ln b^S - \ln a^S\right], \sigma^2 t\right) \frac{1}{s} \\
& \quad - \sum_{k=-\infty}^{\infty} \exp\left(\frac{2\left\{k \ln a^S - [k-1]\ln b^S - \ln s_0\right\}\nu}{\sigma^2}\right) \\
& \qquad\qquad \cdot \phi\left(\ln s; \ln s_0 + \nu t + 2\left\{k \ln a^S - [k-1]\ln b^S\right\}, \sigma^2 t\right) \frac{1}{s}.
\end{aligned} \tag{4.4}$$

Thus, we get the following lemma.

Lemma 9. *The joint probability of $S_t \in I^S$ and $a^S \leq m_t^S \leq M_t^S \leq b^S$ is*

$$\Pr\left(a^S \leq m_t^S \leq M_t^S \leq^S b, S_t \in I^S\right) = \int_{I^S} p_G^S(a,b,s;t)ds,$$

where the trivariate joint probability density $p_G^S\left(a^S, b^S, s; t\right)$ is

$$\begin{aligned}
&p_G^S\left(a^S, b^S, s; t\right) \\
&= \sum_{k=-\infty}^{\infty} \left\{\left[\frac{b^S}{a^S}\right]^k\right\}^{2\mu/\sigma^2-1} \phi\left(\ln\frac{S_t}{s_0}; \ln\left[\frac{b^S}{a^S}\right]^{2k} + \left[\mu - \frac{\sigma^2}{2}\right]t, \sigma^2 t\right)\frac{1}{s} \\
&\quad - \sum_{k=-\infty}^{\infty} \left\{\left[\frac{a^S}{b^S}\right]^k \frac{a^S}{s_0}\right\}^{2\mu/\sigma^2-1} \phi\left(\ln\frac{S_t s_0}{\left[a^S\right]^2}; \ln\left[\frac{a^S}{b^S}\right]^{2k} + \left[\mu - \frac{\sigma^2}{2}\right]t, \sigma^2 t\right)\frac{1}{s}.
\end{aligned}$$

□

4.2 Fourier Series Solution

Applying $X = \ln S$ to Lemma 8, we know the trivariate joint probability density is

$$\begin{aligned}
&p_F^S\left(a^S, b^S, s; t\right) \\
&\doteq \exp\left(\frac{\nu\left[\ln s - \ln s_0\right]}{\sigma^2}\right)\sum_{n=1}^{\infty}\frac{2}{\ln b^S - \ln a^S}\exp\left(-\frac{1}{2}\left\{\frac{\nu^2}{\sigma^2} + \frac{\pi^2\sigma^2 n^2}{\left[\ln b^S - \ln a^S\right]^2}\right\}t\right) \\
&\qquad \cdot \sin\left(\frac{\pi n\left[\ln s_0 - \ln a^S\right]}{\ln b^S - \ln a^S}\right)\sin\left(\frac{\pi n\left[\ln s - \ln a^S\right]}{\ln b^S - \ln a^S}\right)\frac{1}{s}.
\end{aligned} \tag{4.5}$$

Thus, the following lemma holds.

Lemma 10. *The joint probability of $S_t \in I^S$ and $a^S \leq m_t^S \leq M_t^S \leq b^S$ is*

$$\Pr\left(a^S \leq m_t^S \leq M_t^S \leq^S b, S_t \in I^S\right) = \int_{I^S} p_F^S(a,b,s;t)ds,$$

where the trivariate joint probability density $p_F^S\left(a^S, b^S, s; t\right)$ *is*

$$\begin{aligned} & p_F^S\left(a^S, b^S, s; t\right) \\ = & \left[\frac{s}{s_0}\right]^{\nu/\sigma^2} \sum_{n=1}^{\infty} \frac{2}{\ln(b^S/a^S)} \exp\left(-\frac{1}{2}\left\{\left[\frac{\mu}{\sigma}-\frac{\sigma}{2}\right]^2+\frac{\pi^2\sigma^2 n^2}{[\ln(b^S/a^S)]^2}\right\} t\right) \\ & \cdot \sin\left(\frac{\pi n \ln\left(s_0/a^S\right)}{\ln(b^S/a^S)}\right) \sin\left(\frac{\pi n \ln\left(s/a^S\right)}{\ln(b^S/a^S)}\right) \frac{1}{s}. \end{aligned}$$

□

5 Double Barrier Option Pricing

The underlying process $\{S_u \mid t \leq u \leq T\}$ is assumed to be a geometric Brownian motion satisfying

$$dS_u = rS_u du + \sigma S_u dW_u^Q, \tag{5.1}$$

where $\left\{W_t^Q \mid t \leq u \leq T\right\}$ is a standard Brownian motion under the risk-neutral measure Q. Denote its maximum and the minimum, respectively, by

$$M_{[t,u]}^S \doteq \max_{t\leq s\leq u} S_s \quad \text{and} \quad m_{[t,u]}^S \doteq \min_{t\leq s\leq u} S_s. \tag{5.2}$$

The risk-neutral pricing formula for the UODO option with strike K and maturity T is

$$C_t^{UODO} = e^{-r\tau} E_t^Q\left([S_T-K]^+ 1\left(A < m_{[t,T]}^S \leq M_{[t,T]}^S < B\right)\right), \tag{5.3}$$

where $[S_T-K]^+ \doteq \max\{S_T-K, 0\}$. Assume that the constant lower barrier A and the constant upper barrier B satisfy

$$A < s_t \exp(\nu_r\tau) < B, \tag{5.4}$$

where $\tau \doteq T-t$ and

$$\nu_r \doteq r - \frac{\sigma^2}{2}. \tag{5.5}$$

The condition (5.4) is equivalent to the condition (4.3).

Applying Lemma 5 to Eq. (5.1) yields

$$d \ln S_u = \nu_r du + \sigma dW_u^Q. \tag{5.6}$$

Eq. (5.6) implies

$$\ln \frac{S_T}{S_t} = N\left(\nu_r \tau, \sigma^2 \tau\right). \tag{5.7}$$

Set $X_u \doteq \ln S_u$ and $x_t \doteq \ln S_t$. We know from Eq. (5.6) that $\{X_u \mid t \leq u \leq T\}$ is the Brownian motion with drift parameter ν_r and instantaneous variance σ^2. Denote its maximum and minimum, respectively, by

$$M_{[t,u]}^X \doteq \max_{t \leq s \leq u} X_s \quad \text{and} \quad m_{[t,u]}^X \doteq \min_{t \leq s \leq u} X_s. \tag{5.8}$$

Set $k \doteq \ln K$, $\gamma_A \doteq \ln A$, and $\gamma_B \doteq \ln B$. The condition (5.4) becomes

$$\gamma_A < x_t + \nu_r \tau < \gamma_B. \tag{5.9}$$

The risk-neutral pricing formula for the UODO option in Eq. (5.3) becomes

$$C_t^{UODO} = e^{-r\tau} E_t^Q \left(\left[S_t e^{X_T - x_t} - K \right]^+ 1 \left(\gamma_A < m_{[t,T]}^X \leq M_{[t,T]}^X < \gamma_B \right) \right). \tag{5.10}$$

5.1 Gaussian Series Solution

We know from Lemma 7 that the transition probability of $\{X_u\}$ is

$$\begin{aligned} & p_G^Q\left(x_t, t; x_T, T\right) \\ & \doteq \sum_{n=-\infty}^{\infty} \exp\left(\frac{2n\left[\gamma_B - \gamma_A\right] \nu_r}{\sigma^2} \right) \phi\left(x_T; x_t + \nu_r \tau + 2n\left[\gamma_B - \gamma_A\right], \sigma^2 \tau \right) \\ & \quad - \sum_{n=-\infty}^{\infty} \exp\left(\frac{2\left\{ n\gamma_A - [n-1]\gamma_B - x_t \right\} \nu_r}{\sigma^2} \right) \\ & \qquad \cdot \phi\left(x_T; -x_t + \nu_r \tau + 2\left\{ n\gamma_A - [n-1]\gamma_B \right\}, \sigma^2 \tau \right). \end{aligned} \tag{5.11}$$

Putting Eq. (5.11) to Eq. (5.10), we get

$$C_t^{UODO} = S_t I_1 - K e^{-r\tau} I_2, \tag{5.12}$$

where

$$\begin{aligned} I_1 &\doteq e^{-r\tau} \int_k^{\gamma_B} e^{x_T - x_t} p_G^Q \left(x_t, t; x_T, T\right) dx_T, & (5.13) \\ I_2 &\doteq \int_k^{\gamma_B} p_G^Q \left(x_t, t; x_T, T\right) dx_T. & (5.14) \end{aligned}$$

We can easily show that

$$\begin{aligned} & \frac{2\nu_r}{\sigma^2} = \frac{2r}{\sigma^2} - 1, & (5.15) \\ & \int_k^{\gamma_B} \phi\left(x_T; m, \sigma^2\tau\right) dx_T = N\left(\frac{m-k}{\sigma\sqrt{\tau}}\right) - N\left(\frac{m-\gamma_B}{\sigma\sqrt{\tau}}\right). & (5.16) \end{aligned}$$

Applying the transition probability function $p_G^Q\left(x_t, t; x_T, T\right)$ in Eq. (5.11) to Eq. (5.14), we get

$$\begin{aligned} I_2 = & \sum_{n=-\infty}^{\infty} \exp\left(\frac{2n\nu_r}{\sigma^2}\left[\gamma_B - \gamma_A\right]\right) I_{21,n} \\ & - \sum_{n=-\infty}^{\infty} \exp\left(\frac{2\nu_r}{\sigma^2}\left\{n\gamma_A - [n-1]\gamma_B - x_t\right\}\right) I_{22,n}, \end{aligned} \tag{5.17}$$

where

$$\begin{aligned} I_{21,n} &\doteq \int_k^{\gamma_B} \phi\left(x_T; x_t + \nu_r\tau + 2n\left[\gamma_B - \gamma_A\right], \sigma^2\tau\right) dx_T, & (5.18) \\ I_{22,n} &\doteq \int_k^{\gamma_B} \phi\left(x_T; -x_t + \nu_r\tau + 2\left\{n\gamma_A - [n-1]\gamma_B\right\}, \sigma^2\tau\right) dx_T. & (5.19) \end{aligned}$$

Applying Eqs. (5.15) and (5.16) to Eq. (5.18), we get

$$\begin{aligned} I_{21,n} &= N\left(\frac{x_t + \nu_r\tau + 2n\left[\gamma_B - \gamma_A\right] - k}{\sigma\sqrt{\tau}}\right) \\ &\quad - N\left(\frac{x_t + \nu_r\tau + 2n\left[\gamma_B - \gamma_A\right] - \gamma_B}{\sigma\sqrt{\tau}}\right) \\ &= N\left(d_{1,n} - \sigma\sqrt{\tau}\right) - N\left(d_{2,n} - \sigma\sqrt{\tau}\right), \end{aligned} \tag{5.20}$$

where

$$d_{1,n} \doteq \frac{1}{\sigma\sqrt{\tau}} \left\{ \ln \frac{S_t B^{2n}}{K A^{2n}} + \left[r + \frac{\sigma^2}{2} \right] \tau \right\}, \tag{5.21}$$

$$d_{2,n} \doteq \frac{1}{\sigma\sqrt{\tau}} \left\{ \ln \frac{S_t B^{2n}}{B A^{2n}} + \left[r + \frac{\sigma^2}{2} \right] \tau \right\}. \tag{5.22}$$

Applying Eqs. (5.15) and (5.16) to Eqs. (5.19), we get

$$\begin{aligned} I_{22,n} &= N\left(\frac{-x_t + \nu_r \tau + 2\{n\gamma_A - [n-1]\gamma_B\} - k}{\sigma\sqrt{\tau}} \right) \\ &\quad - N\left(\frac{-x_t + \nu_r \tau + 2\{n\gamma_A - [n-1]\gamma_B\} - \gamma_B}{\sigma\sqrt{\tau}} \right) \\ &= N\left(d_{3,n} - \sigma\sqrt{\tau}\right) - N\left(d_{4,n} - \sigma\sqrt{\tau}\right), \end{aligned} \tag{5.23}$$

where

$$d_{3,n} \doteq \frac{1}{\sigma\sqrt{\tau}} \left\{ \ln \frac{A^{2n}}{S_t K B^{2n-2}} + \left[r + \frac{\sigma^2}{2} \right] \tau \right\}, \tag{5.24}$$

$$d_{4,n} \doteq \frac{1}{\sigma\sqrt{\tau}} \left\{ \ln \frac{A^{2n}}{S_t B^{2n-1}} + \left[r + \frac{\sigma^2}{2} \right] \tau \right\}. \tag{5.25}$$

We know from Eqs. (5.17), (5.20), and (5.23) that

$$\begin{aligned} I_2 &= \sum_{n=-\infty}^{\infty} \left[\frac{B^n}{A^n} \right]^{2r/\sigma^2 - 1} \left[N\left(d_{1,n} - \sigma\sqrt{\tau}\right) - N\left(d_{2,n} - \sigma\sqrt{\tau}\right) \right] \\ &\quad - \sum_{n=-\infty}^{\infty} \left[\frac{A^n}{S_t B^{n-1}} \right]^{2r/\sigma^2 - 1} \left[N\left(d_{3,n} - \sigma\sqrt{\tau}\right) - N\left(d_{4,n} - \sigma\sqrt{\tau}\right) \right]. \end{aligned} \tag{5.26}$$

Substituting the transition probability function $p(x_t, t; x_T, T)$ in Eq. (5.11) into Eq. (5.13), we get

$$\begin{aligned} I_1 &= \sum_{n=-\infty}^{\infty} \exp\left(\frac{2n\nu_r}{\sigma^2} [\gamma_B - \gamma_A] \right) I_{11,n} \\ &\quad - \sum_{n=-\infty}^{\infty} \exp\left(\frac{2\nu_r}{\sigma^2} \{n\gamma_A - [n-1]\gamma_B - x_t\} \right) I_{12,n}, \end{aligned} \tag{5.27}$$

where

$$I_{11,n} \doteq \int_k^{\gamma_B} e^{-r\tau} e^{x_T - x_t} \phi\left(x_T; x_t + \nu_r\tau + 2n\left[\gamma_B - \gamma_A\right], \sigma^2\tau\right) dx_T \quad (5.28)$$

$$I_{12,n} \doteq \int_k^{\gamma_B} e^{-r\tau} e^{x_T - x_t} \phi\left(x_T; -x_t + \nu_r\tau + 2\left\{n\gamma_A - [n-1]\gamma_B\right\}, \sigma^2\tau\right) dx_T. \quad (5.29)$$

Applying Lemma 6 to Eq. (5.28), we get

$$I_{11,n} = \exp\left(2n\left[\gamma_B - \gamma_A\right]\right) \cdot \int_k^{\gamma_B} \phi\left(x_T; x_t + \left[r + \frac{\sigma^2}{2}\right]\tau + 2n\left[\gamma_B - \gamma_A\right], \sigma^2 T\right) dx_T. \quad (5.30)$$

Applying Eq. (5.16) to Eqs. (5.30), we get

$$\begin{aligned} \frac{A^{2n}}{B^{2n}} I_{11,n} &= N\left(\frac{1}{\sigma\sqrt{\tau}} B^{2n}\left\{x_t + \nu_r\tau + 2n\left[\gamma_B - \gamma_A\right] - k\right\} + \sigma\sqrt{\tau}\right) \\ &\quad - N\left(\frac{1}{\sigma\sqrt{\tau}}\left\{x_t + \nu_r\tau + 2n\left[\gamma_B - \gamma_A\right] - \gamma_B\right\} + \sigma\sqrt{\tau}\right) \\ &= N\left(d_{1,n}\right) - N\left(d_{2,n}\right). \end{aligned} \quad (5.31)$$

Applying Lemma 6 to Eq. (5.29), we get

$$I_{12,n} = \exp\left(-2x_t + 2\left\{n\gamma_A - [n-1]\gamma_B\right\}\right) \cdot \int_k^{\gamma_B} \phi\left(x_T; -x_t + \left[r + \frac{\sigma^2}{2}\right]\tau + 2\left\{n\gamma_A - [n-1]\gamma_B\right\}, \sigma^2 T\right) dx_T. \quad (5.32)$$

Applying Eq. (5.16) to Eq. (5.32), we get

$$\begin{aligned} \frac{S_t^2 B^{2n-2}}{A^{2n}} I_{12,n} &= N\left(\frac{1}{\sigma\sqrt{\tau}}\left[-x_t + \nu_r\tau + 2\left\{n\gamma_A - [n-1]\gamma_B\right\} - k\right] + \sigma\sqrt{\tau}\right) \\ &\quad - N\left(\frac{1}{\sigma\sqrt{\tau}}\left[-x_t + \nu_r\tau + 2\left\{n\gamma_A - [n-1]\gamma_B\right\} - \tau_B\right] + \sigma\sqrt{\tau}\right) \\ &= N\left(d_{3,n}\right) - N\left(d_{4,n}\right). \end{aligned} \quad (5.33)$$

We know from Eqs. (5.27), (5.31), and (5.33) that

$$
\begin{aligned}
I_1 &= \sum_{n=-\infty}^{\infty}\left[\frac{B^n}{A^n}\right]^{2r/\sigma^2+1}\left[N\left(d_{1,n}\right)-N\left(d_{2,n}\right)\right] \\
&\quad -\sum_{n=-\infty}^{\infty}\left[\frac{A^n}{B^{n-1}S_t}\right]^{2r/\sigma^2+1}\left[N\left(d_{3,n}\right)-N\left(d_{4,n}\right)\right]. \qquad (5.34)
\end{aligned}
$$

Putting Eqs. (5.26) and (5.34) into Eq. (5.12) yields the following lemma.

Lemma 11. *Assume the underlying process $\{S_u \mid t \leq u \leq T\}$ satisfies the stochastic differential equation Eq. (5.1). The Gaussian series solution of the UODO option price at time t with strike price K, maturity T, upper bound B, and lower bound A is*

$$
\begin{aligned}
& C_{t;G}^{UODO} \\
= & \sum_{n=-\infty}^{\infty} S_t\left[\frac{B^n}{A^n}\right]^{2r/\sigma^2+1}\left[N\left(d_{1,n}\right)-N\left(d_{2,n}\right)\right] \\
& -\sum_{n=-\infty}^{\infty} e^{-r\tau}K\left[\frac{B^n}{A^n}\right]^{2r/\sigma^2-1}\left[N\left(d_{1,n}-\sigma\sqrt{\tau}\right)-N\left(d_{2,n}-\sigma\sqrt{\tau}\right)\right] \\
& -\sum_{n=-\infty}^{\infty} S_t\left[\frac{A^n}{S_tB^{n-1}}\right]^{2r/\sigma^2+1}\left[N\left(d_{3,n}\right)-N\left(d_{4,n}\right)\right] \\
& +\sum_{n=-\infty}^{\infty} e^{-r\tau}K\left[\frac{A^n}{S_tB^{n-1}}\right]^{2r/\sigma^2-1}\left[N\left(d_{3,n}-\sigma\sqrt{\tau}\right)-N\left(d_{4,n}-\sigma\sqrt{\tau}\right)\right].
\end{aligned}
$$

□

Readers may refer to Zhang (1998, pp. 309-314) for Lemma 11,

5.2 Fourier Series Solution

We know from Lemma 8 that the transition probability of $\{X_u\}$ is

$$p_F^Q(x_t, t; x_u, u) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n[u-t]} \exp\left(\frac{\nu_r[x_u - x_t]}{\sigma^2}\right) \sin\left(\frac{n\pi[x_u - \gamma_A]}{\gamma_B - \gamma_A}\right), \tag{5.35}$$

where

$$a_n = \frac{2}{\gamma_B - \gamma_A} \sin\left(\frac{n\pi[x_t - \gamma_A]}{\gamma_B - \gamma_A}\right), \tag{5.36}$$

$$\lambda_n \doteq \frac{1}{2}\left\{\frac{\nu_r^2}{\sigma^2} + \frac{n^2\pi^2\sigma^2}{[\gamma_B - \gamma_A]^2}\right\}. \tag{5.37}$$

Applying Eq. (5.35) to Eq. (5.10), we know the risk-neutral pricing formula for the UODO option is

$$C_t^{UODO} = S_t J_1 - K e^{-r\tau} J_2, \tag{5.38}$$

where

$$J_1 \doteq e^{-r\tau} \int_k^b e^{x_T - x_t} p_F^Q(x_t, t; x_T, T)\, dx_T, \tag{5.39}$$

$$J_2 \doteq \int_k^b p_F^Q(x_t, t; x_T, T)\, dx_T. \tag{5.40}$$

Putting Eq. (5.35) into Eq. (5.40), we get

$$\begin{aligned} J_2 &= \sum_{n=1}^{\infty} a_n \exp(-\lambda_n \tau) \exp\left(\frac{\nu_r[\gamma_A - x_t]}{\sigma^2}\right) \\ &\quad \cdot \int_k^b \exp\left(\frac{\nu_r[x_T - \gamma_A]}{\sigma^2}\right) \sin\left(\frac{\pi n[x_T - \gamma_A]}{\gamma_B - \gamma_A}\right) dx_T \\ &= \sum_{n=1}^{\infty} a_n J_{2,n}, \end{aligned} \tag{5.41}$$

where

$$\begin{aligned} J_{2,n} &\doteq \exp(-\lambda_n \tau) \exp\left(\frac{\nu_r[\gamma_A - x_t]}{\sigma^2}\right) \\ &\quad \cdot \int_{k-\gamma_A}^{\gamma_B - \gamma_A} \exp\left(\frac{\nu_r}{\sigma^2} z\right) \sin\left(\frac{\pi n}{\gamma_B - \gamma_A} z\right) dz. \end{aligned} \tag{5.42}$$

The following indefinite integral formula is useful to compute integrals in J_1 and J_2;

$$\int e^{\alpha x} \sin \beta x dx = \frac{1}{\alpha^2 + \beta^2} [\alpha \sin \beta x - \beta \cos \beta x] e^{\alpha x}. \tag{5.43}$$

Applying Eq. (5.43) to Eq. (5.42), we get

$$\begin{aligned} J_{2,n} &= \exp\left(-\frac{1}{2}\left\{\frac{1}{\sigma^2}\left[r - \frac{\sigma^2}{2}\right]^2 + \frac{\pi^2\sigma^2 n^2}{[\gamma_B - \gamma_A]^2}\right\}\tau\right) \\ &\quad \cdot \exp\left(\left[\frac{r}{\sigma^2} - \frac{1}{2}\right][\gamma_A - x_t]\right)\left\{\left[\frac{r}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{\pi^2 n^2}{[\gamma_B - \gamma_A]^2}\right\}^{-1} \\ &\quad \cdot \left[\left\{\frac{\pi n}{[\gamma_B - \gamma_A]}[-1]^{n-1}\right\}\exp\left(\left[\frac{r}{\sigma^2} - \frac{1}{2}\right][\gamma_B - \gamma_A]\right)\right. \\ &\qquad - \left\{\left[\frac{r}{\sigma^2} - \frac{1}{2}\right]\sin\left(\pi n \frac{k - \gamma_A}{\gamma_B - \gamma_A}\right) - \frac{\pi n}{[\gamma_B - \gamma_A]}\cos\left(\pi n \frac{k - \gamma_A}{\gamma_B - \gamma_A}\right)\right\} \\ &\qquad \left.\cdot \exp\left(\left[\frac{r}{\sigma^2} - \frac{1}{2}\right][k - \gamma_A]\right)\right]. \end{aligned} \tag{5.44}$$

Putting Eq. (5.35) into Eq. (5.39), we get

$$\begin{aligned} J_1 &= e^{-r\tau} e^{\gamma_A - x_t} \sum_{n=1}^{\infty} a_n e^{-\lambda_n \tau} \exp\left(\frac{\nu_r[\gamma_A - x_t]}{\sigma^2}\right) \\ &\quad \cdot \int_k^b e^{x_T - \gamma_A} \exp\left(\frac{\nu_r[x_T - \gamma_A]}{\sigma^2}\right) \sin\left(\frac{n\pi[x_T - \gamma_A]}{\gamma_B - \gamma_A}\right) dx_T \\ &= \sum_{n=1}^{\infty} a_n J_{1,n}, \end{aligned} \tag{5.45}$$

where

$$\begin{aligned} J_{1,n} &\doteq e^{-r\tau} \exp(-\lambda_n \tau) e^{\gamma_A - x_t} \exp\left(\frac{\nu_r[\gamma_A - x_t]}{\sigma^2}\right) \\ &\quad \cdot \int_{k-\gamma_A}^{\gamma_B - \gamma_A} \exp\left(\left[\frac{\nu_r}{\sigma^2} + 1\right] z\right) \sin\left(\frac{n\pi}{\gamma_B - \gamma_A} z\right) dz. \end{aligned} \tag{5.46}$$

Applying Eq. (5.43) to Eq. (5.46), we get

$$
\begin{aligned}
J_{1,n} \quad = \quad & \exp\left(-\frac{1}{2}\left\{\frac{1}{\sigma^2}\left[r+\frac{\sigma^2}{2}\right]^2+\frac{\pi^2\sigma^2 n^2}{[\gamma_B-\gamma_A]^2}\right\}\tau\right) \\
& \cdot\exp\left(\left[\frac{r}{\sigma^2}+\frac{1}{2}\right][\gamma_A-x_t]\right)\left\{\left[\frac{r}{\sigma^2}+\frac{1}{2}\right]^2+\frac{\pi^2 n^2}{[\gamma_B-\gamma_A]^2}\right\}^{-1} \\
& \cdot\left[\left\{\frac{\pi n}{[\gamma_B-\gamma_A]}[-1]^{n-1}\right\}\exp\left(\left[\frac{r}{\sigma^2}+\frac{1}{2}\right][\gamma_B-\gamma_A]\right)\right. \\
& \quad -\left\{\left[\frac{r}{\sigma^2}+\frac{1}{2}\right]\sin\left(\pi n\frac{k-\gamma_A}{\gamma_B-\gamma_A}\right)-\frac{\pi n}{[\gamma_B-\gamma_A]}\cos\left(\pi n\frac{k-\gamma_A}{\gamma_B-\gamma_A}\right)\right\} \\
& \quad \left.\cdot\exp\left(\left[\frac{r}{\sigma^2}+\frac{1}{2}\right][k-\gamma_A]\right)\right]. \qquad (5.47)
\end{aligned}
$$

Define a function $J_n(\cdot)$ by

$$
\begin{aligned}
J_n(\eta) \quad \doteq \quad & \exp\left(-\frac{1}{2}\left\{\frac{\eta^2}{\sigma^2}+\frac{\pi^2\sigma^2 n^2}{[\gamma_B-\gamma_A]^2}\right\}\tau\right) \\
& \cdot\exp\left(\frac{\eta}{\sigma^2}[\gamma_A-x_t]\right)\left\{\frac{\eta^2}{\sigma^4}+\frac{\pi^2 n^2}{[\gamma_B-\gamma_A]^2}\right\}^{-1} \\
& \cdot\left[\left\{\frac{\pi n}{[\gamma_B-\gamma_A]}[-1]^{n-1}\right\}\exp\left(\frac{\eta}{\sigma^2}[\gamma_B-\gamma_A]\right)\right. \\
& \quad -\left\{\frac{\eta}{\sigma^2}\sin\left(\pi n\frac{k-\gamma_A}{\gamma_B-\gamma_A}\right)-\frac{\pi n}{[\gamma_B-\gamma_A]}\cos\left(\pi n\frac{k-\gamma_A}{\gamma_B-\gamma_A}\right)\right\} \\
& \quad \left.\cdot\exp\left(\frac{\eta}{\sigma^2}[k-\gamma_A]\right)\right]. \qquad (5.48)
\end{aligned}
$$

We know Eq. (5.44) and Eq. (5.47) that

$$
J_{1,n}=J_n\left(r+\frac{\sigma^2}{2}\right) \quad \text{and} \quad J_{2,n}=J_n\left(r-\frac{\sigma^2}{2}\right). \qquad (5.49)
$$

Substituting Eqs. (5.41), (5.45), and (5.49) into Eq. (5.38), we get the following lemma.

Lemma 12. *Assume the underlying process $\{S_u \mid t \leq u \leq T\}$ satisfies the stochastic differential equation Eq. (5.1). The Fourier series solution of the UODO* *option price at time t with strike price K, maturity T, upper bound B, and lower bound A is*

$$C_{t;F}^{UODO} = \sum_{n=1}^{\infty} \frac{2}{\gamma_B - \gamma_A} \sin\left(\frac{n\pi\,[x_t - \gamma_A]}{\gamma_B - \gamma_A}\right) \cdot \left\{ S_t J_n \left(r + \frac{\sigma^2}{2}\right) - K e^{-r\tau} J_n \left(r - \frac{\sigma^2}{2}\right) \right\}.$$

□

Readers may refer to Zhang (1998, pp. 315-316) for Lemma 11,

5.3 Laplace Transform Solution

Geman and Yor (1996) represent the risk-neutral price C_t^{UODO} using an inverse Laplace transform. We will compare numerically this Laplace transform solution with the Gaussian series solution as well as the Fourier series solution.

Define a stopping time by

$$\tau_{[A,B]}^S \doteq \tau_A^S \wedge \tau_B^S. \tag{5.50}$$

The risk-neutral pricing formula for the UODO option in Eq. (5.3) becomes

$$C_t^{UODO} = e^{-r\tau} E_t^Q \left([S_T - K]^+ 1 \left(\tau_{[A,B]}^S > T \right) \right), \tag{5.51}$$

Set

$$\widehat{W}_s \doteq W_{s+t} - W_t, \quad (s \geq 0), \tag{5.52}$$

$$\widehat{S}_s \doteq \exp\left(\nu_r s + \sigma \widehat{W}_s\right), \quad (s \geq 0), \tag{5.53}$$

$$\tau_{[\alpha,\beta]}^{\widehat{S}} \doteq \tau_\alpha^{\widehat{S}} \wedge \tau_\beta^{\widehat{S}}. \tag{5.54}$$

We know

$$\widehat{S}_{u-t} = \frac{S_u}{S_t}, \quad (u \geq t), \tag{5.55}$$

$$1\left(\tau^{S}_{[A,B]} > T\right) = 1\left(\tau^{S}_{[A,B]} > t\right) 1\left(\tau^{\widehat{S}}_{[A/S_t\,,\,B/S_t]} > T - t\right). \tag{5.56}$$

Since $1\left(\tau^{S}_{[A,B]} > t\right)$ is $\mathcal{F}_t$-measurable,

$$\begin{aligned} & E^Q_t\left([S_T - K]^+ 1\left(\tau^{S}_{[A,B]} > T\right)\right) \\ = \; & 1\left(\tau^{S}_{[A,B]} > t\right) S_t E^Q_t\left(\left[\widehat{S}_\tau - \frac{K}{S_t}\right]^+ 1\left(\tau^{\widehat{S}}_{[A/S_t\,,\,B/S_t]} > \tau\right)\right). \end{aligned} \tag{5.57}$$

Eqs. (5.51) and (5.57) imply

$$C^{UODO}_t = e^{-r\tau} S_t E^Q_t\left(\left[\widehat{S}_\tau - \widehat{K}\right]^+ 1\left(\tau^{\widehat{S}}_{[\widehat{A},\widehat{B}]} > \tau\right)\right), \tag{5.58}$$

where

$$\widehat{K} \doteq \frac{K}{S_t}, \quad \widehat{A} \doteq \frac{A}{S_t}, \quad \widehat{B} \doteq \frac{B}{S_t}. \tag{5.59}$$

Define a function as

$$\varphi^{\widehat{K}}_{\widehat{A},\widehat{B}}(\tau) \doteq E^Q_t\left(\left[\widehat{S}_\tau - \widehat{K}\right]^+ 1\left(\tau^{\widehat{S}}_{[\widehat{A},\widehat{B}]} < \tau\right)\right). \tag{5.60}$$

Clearly,

$$1\left(\tau^{\widehat{S}}_{[\widehat{A},\widehat{B}]} > \tau\right) = 1 - 1\left(\tau^{\widehat{S}}_{[\widehat{A},\widehat{B}]} < \tau\right). \tag{5.61}$$

Eqs. (5.58) and (5.61) imply

$$C^{UODO}_t = S_t \,\mathrm{BS}\left(0, 1, \sigma, \tau, \widehat{K}\right) - e^{-r\tau} S_t\, \varphi^{\widehat{K}}_{\widehat{A},\widehat{B}}(\tau), \tag{5.62}$$

where $\mathrm{BS}\left(0, 1, \sigma, \tau, \widehat{K}\right)$ denotes the fair price at time 0 of a plain-vanilla call option with maturity τ and strike price $\widehat{K}$ written on an underlying which has value 1 at time 0 and volatility σ.

Consider the following Laplace transform of $\varphi_{\widehat{A},\widehat{B}}^{\widehat{K}}(s)$ with respect to time:

$$\Psi(\lambda) \doteq \int_0^\infty e^{-\lambda s} \varphi_{\widehat{A},\widehat{B}}^{\widehat{K}}(s)\, ds. \tag{5.63}$$

Let

$$\widehat{\nu} \doteq \frac{\nu_r}{\sigma^2} = \frac{r}{\sigma^2} - \frac{1}{2}, \tag{5.64}$$

Geman and Yor (1996, p. 371) show that

$$\Psi(\lambda) = \frac{1}{\sigma^2} \Phi\left(\frac{\lambda}{\sigma^2}\right), \tag{5.65}$$

where

$$\Phi(\theta) \doteq \frac{\sinh(\widehat{\mu}\gamma_B)}{\sinh(\widehat{\mu}[\gamma_B - \gamma_A])} g_1 + \frac{\sinh(-\widehat{\mu}\gamma_A)}{\sinh(\widehat{\mu}[\gamma_B - \gamma_A])} g_2, \tag{5.66}$$

$$\widehat{\mu} \doteq \sqrt{2\theta + \widehat{\nu}^2}, \tag{5.67}$$

$$g_1 \doteq \frac{\widehat{K}^{\widehat{\nu}+1-\widehat{\mu}} A^{\mu}}{\widehat{\mu}[\widehat{\mu} - \widehat{\nu}][\widehat{\mu} - \widehat{\nu} - 1]}, \tag{5.68}$$

$$g_2 \doteq 2\left\{\frac{B^{\widehat{\nu}+1}}{\widehat{\mu}^2 - [\widehat{\nu}+1]^2} - \frac{\widehat{K} B^{\widehat{\nu}}}{\widehat{\mu}^2 - \widehat{\nu}^2}\right\} + \frac{B^{-\widehat{\mu}} \widehat{K}^{\widehat{\nu}+1+\widehat{\mu}}}{\widehat{\mu}[\widehat{\mu} + \widehat{\nu}][\widehat{\mu} + \widehat{\nu} + 1]}. \tag{5.69}$$

Eqs. (5.63) and (5.65) imply

$$\varphi_{\widehat{A},\widehat{B}}^{\widehat{K}}(\tau) = \left\{\mathcal{L}^{-1}\Psi\right\}(\tau) = \left\{\mathcal{L}^{-1}\Phi\right\}\left(\sigma^2\tau\right), \tag{5.70}$$

where $\mathcal{L}^{-1}$ is the inverse of the Laplace transform operator.

Combining Eqs. (5.62) and (5.70), we get the following lemma.

Lemma 13. *Assume the underlying process $\{S_u \,|\, t \le u \le T\}$ satisfies the stochastic differential equation Eq. (5.1). The Laplace transform solution of the UODO option price at time t with strike price K, maturity T, upper bound B, and lower bound A is*

$$C_{t;L}^{UODO} = S_t\left[BS\left(0, 1, \sigma, \tau\widehat{K}\right) - e^{-r\tau}\left\{\mathcal{L}^{-1}\Phi\right\}\left(\sigma^2\tau\right)\right],$$

where $\Phi(\cdot)$ is defined in Eq. (5.66). □

An integral formula for the inverse Laplace transform, called the Fourier - Mellin integral, the Bromwich integral (1916), and Mellin's inverse formula, is given by the line integral:

$$\varphi_{\widehat{A},\widehat{B}}^{\widehat{K}}(\tau) = \left\{\mathcal{L}^{-1}\Psi\right\}(\tau) = \frac{1}{2\pi i}\lim_{y\to\infty}\int_{\eta-iy}^{\eta+iy} e^{\lambda\tau}\Psi(\lambda)\,d\lambda \tag{5.71}$$

where η lies to the right of any singularity of the Laplace transform $\Psi(\lambda)$. Practically we calculate the inverse Laplace transform $\left\{\mathcal{L}^{-1}\Psi\right\}(\tau)$ numerically instead of calculating the line integral in Eq. (5.71) analytically. Instead of using a straight line contour, one can use a deformed contour $\Gamma \in \mathbb{C}$ parameterized by

$$\Gamma:\ z(\omega) = x(\omega) + iy(\omega),\ -\infty < \omega < \infty \quad \text{with} \lim_{\omega\to\pm\infty} x(\omega) = 0 \tag{5.72}$$

as long as all the singularities of the integrand lie to the left of Γ. Moreover, practically we choose a contour such that it is symmetric with respect to the real axis. Then the complex conjugacy relation leads to the following representation of integral:

$$\left\{\mathcal{L}^{-1}\Psi\right\}(\tau) = \frac{1}{\pi}\int_0^\infty e^{\tau x(\omega)}\,\mathrm{Im}\left[e^{i\tau y(\omega)}\Psi(z(\omega))\frac{dz}{d\omega}\right](\omega)\,d\omega. \tag{5.73}$$

As a numerical approximation, a composite trapezoidal quadrature rule with interval size $\Delta\omega$ leads to

$$\left\{\mathcal{L}^{-1}\Psi\right\}(\tau) \approx \frac{1}{\pi}{\sum_{j=0}^{N}}' e^{\tau x(\omega_j)}\,\mathrm{Im}\left[e^{i\tau y(\omega_j)}\Psi(z(\omega_j))\frac{dz}{d\omega}(\omega_j)\right]\Delta\omega, \tag{5.74}$$

where $\omega_j = j\Delta\omega, j = 1,\cdots,N$, and ${\sum}'^{N}_{j=0}$ means the summand with $j = 0$ being halved. Of course, equivalently, a composite midpoint rule can be applied. For such deformed contours, numerical solutions will converge very quickly since the exponential factor $e^{\tau x(\omega)}$, with $x(\omega) < 0$ for most of ω, plays a significant role in taming the oscillatory integrand in Eq. (5.73) to decay exponentially fast. Among several others, there have been three types of contours which have been analyzed and popular recently: the Talbot, parabola, and hyperbola types of contours. For details, refer toTalbot (1979), Gavrilyuk and Makarov (2001, 2005, 2007), Sheen, Sloan, and Thomée, (2000, 2003), López-Fernández and Palencia (2004), López-Fernández, Palencia, and Schädle (2006), Lee and Sheen (2009, 2011), Lee, Lee, and Sheen (2013), and Kim and Sheen (2015a). In this study a hyperbola contour of the form is chosen:

$$\begin{aligned} &\Gamma: \ z(\omega) = x(\omega) + iy(\omega), \\ &x(\omega) = \mu(1 - \sin\alpha \cosh\omega), \quad y(\omega) = \mu \cos\alpha \sinh\omega \end{aligned} \tag{5.75}$$

with optimal parameters μ, α, and $\Delta\omega$ chosen as suggested in Weideman (2006, 2010), Weideman and Trefethen (2007), and Kim and Sheen (2015b).

6 Approximation of the UODO Values

We know from Eq. (5.3) that

$$C_t^{UODO} = e^{-r\tau} \int_k^{\gamma_B} [S_T - K]^+ \, p^Q \left(x_t, t; x_T, T\right) dx_T, \tag{6.1}$$

where p^Q is either p_G^Q or p_F^Q. If $\widehat{p}\,(x_t, t; x_T, T)$ is a good approximation of p^Q, then we can approximate C_t^{UODO} as

$$\widehat{C}_t^{UODO} \doteq e^{-r\tau} \int_k^{\gamma_B} [S_T - K]^+ \, \widehat{p}\,(x_t, t; x_T, T) \, dx_T. \tag{6.2}$$

6.1 Gaussian Series solution

Let

$$c_0\left(x_T\right) \doteq \phi\left(x_T; x_t + \nu_r \tau, \sigma^2 \tau\right). \tag{6.3}$$

For $m = 1, 2, \cdots$, let

$$\begin{aligned} c_{2m}\left(x_T\right) \doteq \exp&\left(\frac{2m\left[\gamma_B - \gamma_A\right]\nu_r}{\sigma^2}\right)\phi\left(x_T; x_t + \nu_r\tau + 2m\left[\gamma_B - \gamma_A\right], \sigma^2\tau\right) \\ &+ \exp\left(\frac{-2m\left[\gamma_B - \gamma_A\right]\nu_r}{\sigma^2}\right)\phi\left(x_T; x_t + \nu_r\tau - 2m\left[\gamma_B - \gamma_A\right], \sigma^2\tau\right), \end{aligned} \tag{6.4}$$

and

$$\begin{aligned} c_{2m-1}\left(x_T\right) \doteq \exp&\left(\frac{2\left\{m\gamma_B - [m-1]\gamma_A - x_t\right\}\nu_r}{\sigma^2}\right) \\ &\cdot \phi\left(x_T; x_t + \nu_r\tau + 2[m-1]\left[\gamma_B - \gamma_A\right] + 2\gamma_B, \sigma^2\tau\right) \\ &+ \exp\left(\frac{2\left\{[-m+1]\gamma_B + m\gamma_A - x_t\right\}\nu_r}{\sigma^2}\right) \\ &\cdot \phi\left(x_T; x_t + \nu_r\tau - 2m\left[\gamma_B - \gamma_A\right] + 2\gamma_B, \sigma^2\tau\right). \end{aligned} \tag{6.5}$$

Eq. (5.11) implies

$$p_G^Q\left(x_t, t; x_T, T\right) = \sum_{n=0}^{\infty}[-1]^n c_n\left(x_T\right). \tag{6.6}$$

Define an approximation of $p_G^Q\left(x_t, t; x_T, T\right)$ by

$$\widehat{p}_G^{(N)}\left(x_t, t; x_T, T\right) \doteq \sum_{n=0}^{N}[-1]^n c_n\left(x_T\right). \tag{6.7}$$

It is trivial that $\sum_{n=0}^{\infty}[-1]^n c_n\left(x_T\right)$ is an alternating series. We can show that $\left\{c_n\left(x_T\right)\right\}$ is positive and decreasing for each $x_T \in [\gamma_A, \gamma_B]$, and that

$$\lim_{n\to} c_n\left(x_T\right) = 0, \quad \left(x_T \in [\gamma_A, \gamma_B]\right). \tag{6.8}$$

Thus, the alternating series test implies $\sum_{n=0}^{\infty}[-1]^n c_n(x_T)$ converges, and

$$\left|p_G^Q(x_t,t;x_T,T)-\widehat{p}_G^{(N)}(x_t,t;x_T,T)\right|=\left|\sum_{n=N+1}^{\infty}[-1]^n c_n(x_T)\right|\le c_{N+1}(x_T). \tag{6.9}$$

Define an approximation of $C_{t;G}^{UODO}$ by

$$\widehat{C}_{t;G}^{(N)}\doteq e^{-r\tau}\int_k^{\gamma_B}[S_T-K]^+\,\widehat{p}_G^{(N)}(x_t,t;x_T,T)\,dx_T. \tag{6.10}$$

We know

$$\begin{aligned}
&\left|C_{t;G}^{UODO}-\widehat{C}_{t;G}^{(N)}\right|\\
&\le\left|e^{-r\tau}\int_k^{\gamma_B}[S_T-K]^+\left[p^Q(x_t,t;x_T,T)-\widehat{p}_G^{(N)}(x_t,t;x_T,T)\right]dx_T\right|\\
&\le e^{-r\tau}\int_k^{\gamma_B}[S_T-K]^+\left|p^Q(x_t,t;x_T,T)-\widehat{p}_G^{(N)}(x_t,t;x_T,T)\right|dx_T\\
&\le e^{-r\tau}[B-K]\int_k^{\gamma_B}c_{N+1}(x_T)\,dx_T\\
&\le e^{-r\tau}[B-K]\,[\gamma_B-k]\max_{k\le x_T\le\gamma_B}c_{N+1}(x_T).
\end{aligned} \tag{6.11}$$

Eqs. (5.9) and (6.4) imply

$$\begin{aligned}
&\max_{k\le x_T\le\gamma_B}c_{2m}(x_T)\\
&=\exp\left(\frac{2m\,[\gamma_B-\gamma_A]\,\nu_r}{\sigma^2}\right)\phi\left(\gamma_B;x_t+\nu_r\tau+2m\,[\gamma_B-\gamma_A]\,,\sigma^2\tau\right)\\
&\quad+\exp\left(\frac{-2m\,[\gamma_B-\gamma_A]\,\nu_r}{\sigma^2}\right)\phi\left(k;x_t+\nu_r\tau-2m\,[\gamma_B-\gamma_A]\,,\sigma^2\tau\right).
\end{aligned} \tag{6.12}$$

Thus, for large m,

$$\begin{aligned}
&\max_{k\le x_T\le\gamma_B}c_{2m}(x_T)\\
&\approx\exp\left(\frac{2m\,[\gamma_B-\gamma_A]\,\nu_r}{\sigma^2}\right)\phi\left(\gamma_B;x_t+\nu_r\tau+2m\,[\gamma_B-\gamma_A]\,,\sigma^2\tau\right).
\end{aligned} \tag{6.13}$$

Eqs. (5.9) and (6.5) imply

$$\begin{aligned}
&\max_{k \le x_T \le \gamma_B} c_{2m-1}(x_T) \\
&= \exp\left(\frac{2\{m\gamma_B - [m-1]\gamma_A - x_t\}\nu_r}{\sigma^2}\right) \\
&\quad \cdot \phi\left(\gamma_B; x_t + \nu_r\tau + 2[m-1]\left[\gamma_B - \gamma_A\right] + 2\gamma_B, \sigma^2\tau\right) \\
&+ \exp\left(\frac{2\{[-m+1]\gamma_B + m\gamma_A - x_t\}\nu_r}{\sigma^2}\right) \\
&\quad \cdot \phi\left(k; x_t + \nu_r\tau - 2m\left[\gamma_B - \gamma_A\right] + 2\gamma_B, \sigma^2\tau\right). \qquad (6.14)
\end{aligned}$$

Thus, for large m,

$$\begin{aligned}
&\max_{k \le x_T \le \gamma_B} c_{2m-1}(x_T) \\
&\approx \exp\left(\frac{2\{m\gamma_B - [m-1]\gamma_A - x_t\}\nu_r}{\sigma^2}\right) \\
&\quad \cdot \phi\left(\gamma_B; x_t + \nu_r\tau + 2[m-1]\left[\gamma_B - \gamma_A\right] + 2\gamma_B, \sigma^2\tau\right) \\
&\approx \exp\left(\frac{[2m-1]\left[\gamma_B - \gamma_A\right]\nu_r}{\sigma^2}\right) \phi\left(\gamma_B; x_t + \nu_r\tau + [2m-1]\left[\gamma_B - \gamma_A\right], \sigma^2\tau\right). \qquad (6.15)
\end{aligned}$$

It should be noted that

$$\begin{aligned}
c_1(x_T) \doteq\ &\exp\left(\frac{2\{\gamma_B - x_t\}\nu_r}{\sigma^2}\right) \phi\left(x_T; x_t + \nu_r\tau + 2\gamma_B, \sigma^2\tau\right) \\
&+ \exp\left(\frac{2\{\gamma_A - x_t\}\nu_r}{\sigma^2}\right) \phi\left(x_T; x_t + \nu_r\tau + 2\gamma_A, \sigma^2\tau\right). \qquad (6.16)
\end{aligned}$$

Thus, the approximation $\max\limits_{k \le x_T \le \gamma_B} c_{2m-1}(x_T)$ in Eq. (6.15) is not eligible for $\max\limits_{k \le x_T \le \gamma_B} c_1(x_T)$. We know from Eqs. (6.13) and (6.15) that, for large n,

$$\begin{aligned}
&\max_{k \le x_T \le \gamma_B} c_n(x_T) \\
&\approx \exp\left(\frac{n\left[\gamma_B - \gamma_A\right]\nu_r}{\sigma^2}\right) \phi\left(\gamma_B; x_t + \nu_r\tau + n\left[\gamma_B - \gamma_A\right], \sigma^2\tau\right) \\
&= \frac{1}{\sigma\sqrt{2\pi\tau}} \exp\left(-\frac{\{x_t + \nu_r\tau + n\left[\gamma_B - \gamma_A\right] - \gamma_B\}^2}{2\sigma^2\tau}\right) \\
&\quad \cdot \exp\left(-\frac{\nu\{2[x_t + \nu_r\tau] - 2\gamma_B - \nu_r\tau\}}{2\sigma^2}\right). \qquad (6.17)
\end{aligned}$$

Set

$$e_G^{(N)} \doteq \alpha_G \exp\left(-\frac{\{x_t + \nu_r\tau + [N+1][\gamma_B - \gamma_A] - \gamma_B\}^2}{2\sigma^2\tau}\right), \tag{6.18}$$

where

$$\alpha_G \doteq \frac{e^{-r\tau}[B-K][\gamma_B - k]}{\sigma\sqrt{2\pi\tau}} \exp\left(-\frac{\nu[2x_t + \nu_r\tau - 2\gamma_B]}{2\sigma^2}\right). \tag{6.19}$$

Eqs. (6.11), (6.16), and (6.17) imply the following theorem.

Theorem 2. *For large N, $e_G^{(N)}$ is an approximate error bound of an approximate value $\widehat{C}_{t;G}^{(N)}$ for the true value $C_{t;G}^{UODO}$. Moreover, if $N > N_G^*$, where*

$$N_G^* \doteq \max\left\{\frac{1}{\gamma_B - \gamma_A}\left[\gamma_B - x_t + \sqrt{-2\sigma^2\tau \ln\frac{\epsilon}{\alpha_G}}\,\right] - 1, 1\right\},$$

then $e_G^{(N)}$ is less than a small $\epsilon > 0$.

Let

$$\begin{aligned} C_0^G \doteq\ & S_t\left[N(d_{1,0}) - N(d_{2,0})\right] \\ & - e^{-r\tau}K\left[N\left(d_{1,0} - \sigma\sqrt{\tau}\right) - N\left(d_{2,0} - \sigma\sqrt{\tau}\right)\right]. \end{aligned} \tag{6.20}$$

For $m = 1, 2, \cdots$, let

$$\begin{aligned} C_{2m}^G \doteq\ & \left\{S_t\left[\frac{B^m}{A^m}\right]^{2r/\sigma^2+1}\left[N(d_{1,m}) - N(d_{2,m})\right]\right. \\ & \left. -e^{-r\tau}K\left[\frac{B^m}{A^m}\right]^{2r/\sigma^2-1}\left[N\left(d_{1,m} - \sigma\sqrt{\tau}\right) - N\left(d_{2,m} - \sigma\sqrt{\tau}\right)\right]\right\} \\ & + \left\{S_t\left[\frac{B^{-m}}{A^{-m}}\right]^{2r/\sigma^2+1}\left[N(d_{1,-m}) - N(d_{2,-m})\right]\right. \\ & \left. -e^{-r\tau}K\left[\frac{B^{-m}}{A^{-m}}\right]^{2r/\sigma^2-1}\left[N\left(d_{1,-m} - \sigma\sqrt{\tau}\right) - N\left(d_{2,-m} - \sigma\sqrt{\tau}\right)\right]\right\}, \end{aligned} \tag{6.21}$$

and

$$
\begin{aligned}
C^G_{2m-1} \doteq & \left\{ S_t \left[\frac{A}{S_t}\frac{A^m}{B^m}\right]^{2r/\sigma^2+1} \left[N\left(d_{3,m}\right) - N\left(d_{4,m}\right)\right] \right. \\
& \left. -e^{-r\tau}K\left[\frac{A}{S_t}\frac{A^m}{B^m}\right]^{2r/\sigma^2-1} \left[N\left(d_{3,m}-\sigma\sqrt{\tau}\right) - N\left(d_{4,m}-\sigma\sqrt{\tau}\right)\right] \right\} \\
& + \left\{ S_t \left[\frac{A}{S_t}\frac{A^{1-m}}{B^{1-m}}\right]^{2r/\sigma^2+1} \left[N\left(d_{3,1-m}\right) - N\left(d_{4,1-m}\right)\right] \right. \\
& \left. -e^{-r\tau}K\left[\frac{A}{S_t}\frac{A^{1-m}}{B^{1-m}}\right]^{2r/\sigma^2-1} \left[N\left(d_{3,1-m}-\sigma\sqrt{\tau}\right) - N\left(d_{4,1-m}-\sigma\sqrt{\tau}\right)\right] \right\}.
\end{aligned}
\tag{6.22}
$$

We know from Eqs. (6.7) and (6.10) that

$$
\widehat{C}^{(N)}_{t;G} = \sum_{n=0}^{N}[-1]^n C^G_n. \tag{6.23}
$$

6.2 Fourier Series solution

Define an approximation of $p^Q_F\left(x_t, t; x_T, T\right)$ by

$$
\widehat{p}^{(N)}_F\left(x_t, t; x_T, T\right) \doteq \sum_{n=1}^{N} d_n\left(x_T\right), \tag{6.24}
$$

where

$$
\begin{aligned}
d_n\left(x_T\right) \doteq & \frac{2}{\gamma_B-\gamma_A}\exp\left(\frac{\nu_r\left[x_T-x_t\right]}{\sigma^2}\right)e^{-\lambda_n\tau} \\
& \cdot \sin\left(\frac{n\pi\left[x_T-\gamma_A\right]}{\gamma_B-\gamma_A}\right)\sin\left(\frac{n\pi\left[x_t-\gamma_A\right]}{\gamma_B-\gamma_A}\right).
\end{aligned}
\tag{6.25}
$$

It is clear that

$$
\left|d_n\left(x_T\right)\right| \leq \frac{2}{\gamma_B-\gamma_A}\exp\left(\frac{\nu_r\left[x_T-x_t\right]}{\sigma^2}\right)e^{-\lambda_n\tau}, \quad (n = 1, 2, \cdots). \tag{6.26}
$$

Applying the ratio test and the comparison test, we can show $\sum\limits_{n=1}^{\infty} d_n\left(x_T\right)$ converges absolutely. Define an approximation of $C^{UODO}_{t;F}$ by

$$
\widehat{C}^{(N)}_{t;F} \doteq e^{-r\tau}\int_k^{\gamma_B}\left[S_T-K\right]^+ \widehat{p}^{(N)}_F\left(x_t, t; x_T, T\right)dx_T. \tag{6.27}
$$

Thus,

$$\widehat{C}^{(N)}_{t;F} \doteq e^{-r\tau} \int_k^{\gamma_B} [S_T - K]^+ \sum_{n=1}^{N} d_n(x_T)\, dx_T. \tag{6.28}$$

We know from Lemma 12 and Eq. (6.28) that

$$\begin{aligned}\widehat{C}^{(N)}_{t;F} = & \sum_{n=1}^{N} \frac{2}{\gamma_B - \gamma_A} \sin\left(\frac{n\pi\, [x_t - \gamma_A]}{\gamma_B - \gamma_A}\right) \\ & \cdot \left\{ S_t J_n\left(r + \frac{\sigma^2}{2}\right) - K e^{-r\tau} J_n\left(r - \frac{\sigma^2}{2}\right)\right\}.\end{aligned} \tag{6.29}$$

Eqs. (5.35), (6.24), and 6.25 imply

$$p^Q_F(x_t, t; x_T, T) = \sum_{n=1}^{\infty} d_n(x_T). \tag{6.30}$$

Thus,

$$C^{UODO}_{t;F} = e^{-r\tau} \int_k^{\gamma_B} [S_T - K]^+ \sum_{n=1}^{\infty} d_n(x_T)\, dx_T. \tag{6.31}$$

Eqs. (6.28) and (6.31) imply

$$\begin{aligned}&\left| C^{UODO}_{t;F} - \widehat{C}^{(N)}_{t;F} \right| \\ &= \left| e^{-r\tau} \int_k^{\gamma_B} [S_T - K]^+ \sum_{n=N+1}^{\infty} d_n(x_T)\, dx_T \right| \\ &\le e^{-r\tau}[B - K] \sum_{n=N+1}^{\infty} \int_k^{\gamma_B} |d_n(x_T)|\, dx_T \\ &\le e^{-r\tau}[B - K] \frac{2}{\gamma_B - \gamma_A} \int_k^{\gamma_B} \exp\left(\frac{\nu_r\, [x_T - x_t]}{\sigma^2}\right) dx_T \sum_{n=N+1}^{\infty} e^{-\lambda_n \tau},\end{aligned} \tag{6.32}$$

where the last inequality holds by Eq. (6.26). Clearly,

$$\begin{aligned}&\int_k^{\gamma_B} \exp\left(\frac{\nu_r\, [x_T - x_t]}{\sigma^2}\right) dx_T \\ &= \begin{cases} \frac{\sigma^2}{\nu_r} \exp\left(\frac{-\nu_r x_t}{\sigma^2}\right) \left\{\exp\left(\frac{\nu_r \gamma_B}{\sigma^2}\right) - \exp\left(\frac{\nu_r k}{\sigma^2}\right)\right\}, & (\nu_r \neq 0), \\ \gamma_B - k, & (\nu_r = 0). \end{cases}\end{aligned} \tag{6.33}$$

We know

$$
\begin{aligned}
&\sum_{n=N+1}^{\infty} e^{-\lambda_n \tau} \\
&= \exp\left(-\frac{\nu_r^2}{2\sigma^2}\tau\right) \sum_{n=N+1}^{\infty} \exp\left(-\frac{\pi^2\sigma^2 n^2}{2\left[\gamma_B-\gamma_A\right]^2}\tau\right) \\
&= \exp\left(-\frac{\nu_r^2}{2\sigma^2}\tau\right) \exp\left(-\frac{\pi^2\sigma^2[N+1]^2}{2\left[\gamma_B-\gamma_A\right]^2}\tau\right) \sum_{k=0}^{\infty} \exp\left(-\frac{\pi^2\sigma^2[2N+2+k]k}{2\left|\gamma_B-\gamma_A\right|^2}\tau\right) \\
&\leq \exp\left(-\frac{\nu_r^2}{2\sigma^2}\tau\right) \exp\left(-\frac{\pi^2\sigma^2[N+1]^2}{2\left[\gamma_B-\gamma_A\right]^2}\tau\right) \sum_{k=0}^{\infty} \exp\left(-\frac{\pi^2\sigma^2 \cdot Nk}{\left|\gamma_B-\gamma_A\right|^2}\tau\right) \\
&\leq \exp\left(-\frac{\nu_r^2}{2\sigma^2}\tau\right) \exp\left(-\frac{\pi^2\sigma^2[N+1]^2}{2\left[\gamma_B-\gamma_A\right]^2}\tau\right) \left[1-\exp\left(-\frac{\pi^2\sigma^2 N}{\left|\gamma_B-\gamma_A\right|^2}\tau\right)\right]^{-1}.
\end{aligned}
\tag{6.34}
$$

If $N \geq \left[\gamma_B-\gamma_A\right]^2 / \left[\pi^2\sigma^2\tau\right]$, then Eq. (6.34) implies

$$
\sum_{n=N+1}^{\infty} e^{-\lambda_n \tau} \leq \exp\left(-\frac{\nu_r^2}{2\sigma^2}\tau\right) \exp\left(-\frac{\pi^2\sigma^2[N+1]^2}{2\left[\gamma_B-\gamma_A\right]^2}\tau\right) \frac{e}{e-1}. \tag{6.35}
$$

If $\nu_r \neq 0$, then let

$$
\begin{aligned}
\alpha_F \doteq\ & \frac{2e[B-K]\sigma^2}{[e-1]\left[\gamma_B-\gamma_A\right]} \exp\left(-\left\{r\tau + \frac{\nu_r x_t}{\sigma^2} + \frac{\nu_r^2}{2\sigma^2}\tau\right\}\right) \\
& \cdot \frac{1}{\nu_r}\left\{\exp\left(\frac{\nu_r \gamma_B}{\sigma^2}\right) - \exp\left(\frac{\nu_r k}{\sigma^2}\right)\right\}.
\end{aligned}
\tag{6.36}
$$

If $\nu_r = 0$, then let

$$
\alpha_F \doteq \frac{2e[B-K]\left[\gamma_B-k\right]}{[e-1]\left[\gamma_B-\gamma_A\right]} \exp\left(-r\tau\right). \tag{6.37}
$$

Also, let

$$
e_F^{(N)} \doteq \alpha_F \exp\left(-\frac{\pi^2\sigma^2[N+1]^2}{2\left[\gamma_B-\gamma_A\right]^2}\tau\right), \tag{6.38}
$$

Eqs. (6.32), (6.33) and (6.35) imply

$$
\left|C_{t;F}^{UODO} - \widehat{C}_{t;F}^{(N)}\right| \leq e_F^{(N)}. \tag{6.39}
$$

Eqs. (6.38) and (6.39) imply the following theorem.

Theorem 3. *For large N, $e_F^{(N)}$ is an error bound of an approximate value $\widehat{C}_{t;F}^{(N)}$ for the true value $C_{t;F}^{UODO}$. Moreover, if*

$$N \geq \max\left\{ \frac{[\gamma_B - \gamma_A]^2}{\pi^2\sigma^2}, \frac{\gamma_B - \gamma_A}{\pi\sigma}\sqrt{-\frac{2}{\tau}\ln\frac{\epsilon}{\alpha_F}} - 1 \right\},$$

then $e_F^{(N)}$ is less than a small $\epsilon > 0$.

We know from Lemma 12 that

$$\begin{aligned}\widehat{C}_{t;F}^{(N)} &= \sum_{n=1}^{N} \frac{2}{\gamma_B - \gamma_A} \sin\left(\frac{n\pi\,[x_t - \gamma_A]}{\gamma_B - \gamma_A}\right) \\ &\quad \cdot \left\{ S_t J_n\left(r + \frac{\sigma^2}{2}\right) - Ke^{-r\tau} J_n\left(r - \frac{\sigma^2}{2}\right)\right\}.\end{aligned}$$

7 Numerical Examples

We compute the fair value of an UODO option in four different settings as in Table 1, which was presented by Geman and Yor (1996). The standard deviation of Monte Carlo estimates is computed on a sample of 200 evaluations, each evaluation being performed on 5000 Monte Carlo paths.

7.1 Gaussian Series solution

In Case 1, the iteration stopping number with $\epsilon = 10^{-12}$ is 2.266. Thus, we let $N_G^* = 3$. The Gaussian series solution is in Table 2. In the table, the term *No. CDF's* means the number of normal cumulative distribution functions to be calculated. In this case, 28 normal cumulative distribution functions should be calculated to have an error less than 10^{-12} in absolute value.

Table 1: Geman and Yor's Examples

	Case 1	Case 2	Case 3	Case 4
(t, T, S_t)	$(0, 1, 2)$	$(0, 1, 2)$	$(0, 1, 2)$	$(0, 1/12, 2.4)$
(σ, r, K)	$(0.2, 0.02, 2)$	$(0.5, 0.05, 2)$	$(0.5, 0.05, 1.75)$	$(0.2, 0.02, 2)$
(A, B)	$(1.5, 2.5)$	$(1.5, 3)$	$(1, 3)$	$(1.5, 2.5)$
$\nu_r = r - \frac{\sigma^2}{2}$	0	-0.075	- 0.075	0
BS $\left(0, 1, \sigma, \tau, \widehat{K}\right)$	0.0892	0.2179	0.27646	0.1681
$e^{-r\tau}\left\{\mathcal{L}^{-1}\Psi\right\}(\tau)$	0.0687	0.2090	0.23838	0.23015
Laplace Solution	0.0411	0.0178	0.07615	0.17321
Gaussian Solution	0.041089	0.017856	0.076172	
Monte Carlo	0.0425	0.0191	0.0772	0.1739
(st. dev)	(0.003)	(0.003)	(0.003)	(0.008)

Table 2: Gaussian Series Solution of Case 1

n	No. CDF's	Gaussian Solution	Error
0	4	0.077,810,462	-0.0367
1	12	0.041,075,978	1.3E-5
2	20	0.041,088,551	-1E-10
3	28	0.041,088,550	0
4	36	0.041,088,550	0

In Case 2, the iteration stopping number with $\epsilon = 10^{-12}$ is 4.819. Thus, we let $N_G^* = 5$. The Gaussian series solution is in Table 3. In this case, 44 normal cumulative distribution functions should be calculated to have an error less than 10^{-12} in absolute value.

In Case 3, the iteration stopping number with $\epsilon = 10^{-12}$ is 2.7028. Thus, we let $N_G^* = 3$. The Gaussian series solution is in Table 4. In this case, 28 normal cumulative distribution functions should be calculated to have an error less than 10^{-12} in absolute value.

In Case 4, the iteration stopping number with $\epsilon = 10^{-12}$ is 1. Thus, we let $N_G^* = 1$. The Gaussian series solution is in Table 5. In this case, 12 normal

Table 3: Gaussian Series Solution of Case 2

n	No. CDF's	Gaussian Solution	Error
0	4	0.111,175,114	-0.0933
1	12	0.005,646,212	0.0122
2	20	0.017,962,639	-0.0001
3	28	0.017,856,332	6.9E-7
4	36	0.017,857,021	-7.1E-12
5	44	0.017,857,021	2.2E-14
6	52	0.017,857,021	0
7	60	0.017,857,021	0

Table 4: Gaussian Series Solution of Case 3

n	No. CDF's	Gaussian Solution	Error
0	4	0.188,218,320	-0.1120
1	12	0.076,035,418	0.0001
2	20	0.076,172,370	-8.3E-8
3	28	0.076,172,287	-1E-15
4	36	0.076,172,287	0
5	44	0.076,172,287	0

cumulative distribution functions should be calculated to have an error less than 10^{-12} in absolute value.

Table 5: Gaussian Series Solution of Case 4

n	No. CDF's	Gaussian Solution	Error
0	4	0.262,731,403	-0.0999
1	12	0.162,824,119	0
2	20	0.162,824,119	0

7.2 Fourier Series solution

In Case 1, the iteration stopping number with $\epsilon = 10^{-12}$ is 5.0010. Thus, we let $N_F^* = 6$. The Fourier series solution is in Table 6. From this table, we know that $N_F^* = 5$ is enough to have an error less than 10^{-12} in absolute value.

Table 6: Fourier Series Solution of Case 1

n	Fourier Solution	Error
1	0.038,805,445	0.0023
2	0.041,199,030	-0.0001
3	0.041,088,893	-3.4E-7
4	0.041,088,550	1.9E-10
5	0.041,088,550	6.4E-14
6	0.041,088,550	0
7	0.041,088,550	0

In Case 2, the iteration stopping number with $\epsilon = 10^{-12}$ is 2.310. Thus, we let $N_F^* = 3$. The Fourier series solution is in Table 7.

Table 7: Fourier Series Solution of Case 2

n	Fourier Solution	Error
1	0.017,862,164	-5.1E-6
2	0.017,857,021	-1.3E-11
3	0.017,857,021	0
4	0.017,857,021	0

In Case 3, the iteration stopping number with $\epsilon = 10^{-12}$ is 4.252. Thus, we let $N_F^* = 5$. The Fourier series solution is in Table 8. From this table, we know that $N_F^* = 4$ is enough to have an error less than 10^{-12} in absolute value.

Table 8: Fourier Series Solution of Case 3

n	Fourier Solution	Error
1	0.072,386,928	0.0038
2	0.076,180,974	-8.7E-6
3	0.076,172,301	1.3E-8
4	0.076,172,287	-4.4E-13
5	0.076,172,287	0
6	0.076,172,287	0

In Case 4, the iteration stopping number with $\epsilon = 10^{-12}$ is 19.795. Thus, we let $N_F^* = 20$. The Fourier series solution is in Table 9. From this table, we know that $N_F^* = 19$ is enough to have an error less than 10^{-12} in absolute value.

Table 9: Fourier Series Solution of Case 4

n	Fourier Solution	Error
1	0.020,036,024	0.1428
2	0.068,650,707	0.0942
3	0.116,212,209	0.0466
10	0.162,818,805	5.3E-6
15	0.162,824,121	- 1.7E-9
18	0.162,824,119	- 2.3E-12
19	0.162,824,119	- 1.8E-13
20	0.162,824,119	-1.2E-14
21	0.162,824,119	0
22	0.162,824,119	0

7.3 Laplace transform Solution

In Case 1, the Black-Scholes price is

$$\mathrm{BS}\,(0, 1, 0.2, 1, 2/2) = 8.916, 037, 279 \times 10^{-2}. \tag{7.1}$$

Denote by n_{CQP} the number N of quadrature points on the deformed contour (5.75) in the numerical Laplace inversion formula (5.74). We remarks that the numerical solutions by this Laplace inversion algorithms are calculated using quadruple precision. The Laplace transform solution is in Table 10.

In Case 2, the Black-Scholes price is

$$\mathrm{BS}\,(0, 1, 0.5, 1, 2/2) = 0.217, 926, 042. \tag{7.2}$$

The Laplace transform solution is in Table 11.

Table 10: Laplace Transform Solution of Case 1

n	n_{CQP}	$e^{-r\tau}\left\{\mathcal{L}^{-1}\Psi\right\}(\tau)$	Laplace Solution
1	10	6.861,609,798E-2	4.108,854,961E-2
2	13	6.861,609,757E-2	4.108,855,044E-2
3	16	6.861,609,757E-2	4.108,855,044E-2
4	19	6.861,609,757E-2	4.108,855,044E-2
5	22	6.861,609,757E-2	4.108,855,044E-2

Table 11: Laplace Transform Solution of Case 2

n	n_{CQP}	$e^{-r\tau}\left\{\mathcal{L}^{-1}\Psi\right\}(\tau)$	Laplace Solution
1	10	0.208,997,533	1.785,701,821E-2
2	13	0.208,997,532	1.785,702,099E-2
3	16	0.208,997,532	1.785,702,099E-2
4	19	0.208,997,532	1.785,702,099E-2
5	22	0.208,997,532	1.785,702,099E-2

In Case 3, the Black-Scholes price is

$$\mathrm{BS}\left(0,1,0.5,1,1.75/2\right)=0.276,458,376. \tag{7.3}$$

The Laplace transform solution is in Table 12.

Table 12: Laplace Transform Solution of Case 3

n	n_{CQP}	$e^{-r\tau}\left\{\mathcal{L}^{-1}\Psi\right\}(\tau)$	Laplace Solution
1	10	0.237,611,543	7.769,366,677E-2
2	13	0.237,611,542	7.769,366,938E-2
3	16	0.237,611,542	7.769,366,939E-2
4	19	0.237,611,542	7.769,366,939E-2
5	22	0.237,611,542	7.769,366,939E-2

In Case 4, the Black-Scholes price is

$$\mathrm{BS}\left(0,1,0.2,1/12,2/2.4\right)=0.168,064,640. \tag{7.4}$$

The Laplace transform solution is in Table 13.

Table 13: Laplace Transform Solution of Case 4

n	n_{CQP}	$e^{-r\tau}\left\{\mathcal{L}^{-1}\Psi\right\}(\tau)$	Laplace Solution
1	10	0.236,555,390	0.166,799,747
2	13	0.236,555,390	0.166,799,747
3	16	0.236,555,390	0.166,799,747
4	19	0.236,555,390	0.166,799,747
5	22	0.236,555,390	0.166,799,747

8 Conclusion

The trivariate joint probability density of Brownian motion and its maximum and minimum is represented by two infinite product forms. The two infinite product formulas are computationally more efficient than the previous known Gaussian and Fourier infinite series formulas. A general form of the joint probability density is proposed, which is a linear combination of the two infinite product formulas as well as Gaussian and Fourier infinite series formulas. Using the trivariate joint probability densities, a double barrier option can be priced under the Black-Scholes environment. In practice it is necessary to use some approximate prices of a double barrier option, for the joint probability density is represented by infinite series or infinite products. In this paper we present Gaussian series and Fourier series approximations of an UODO option, their error bounds, and stopping rules for approximations.

Numerical examples of calculating some UODO option values are presented to show usefulness of the approximations. For that purpose we also calculate them using the inverse Laplace transform. From the experiments we can conclude that either the Gaussian approximation or the Fourier approximation is more accurate than the inverse Laplace transform solution, and that the Fourier approximation is much faster than Gaussian one. However, when the tenor τ is small, the Gaussian approximation converges very fast.

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